GRADUAL CLASSICAL LOGIC FOR ATTRIBUTED OBJECTS - EXTENDED IN RE-PRESENTATION

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Abstract. Our understanding about things is conceptual. By stating that we reason about objects, it is in fact not the objects but concepts referring to them that we manipulate. Now, so long just as we acknowledge infinitely extending notions such as space, time, size, colour, etc, - in short, any reasonable quality - into which an object is subjected, it becomes infeasible to affirm atomicity in the concept referring to the object. However, formal/symbolic logics typically presume atomic entities upon which other expressions are built. Can we reflect our intuition about the concept onto formal/symbolic logics at all? I assure that we can, but the usual perspective about the atomicity needs inspected. In this work, I present *gradual logic* which materialises the observation that we cannot tell apart whether a so-regarded atomic entity is atomic or is just atomic enough not to be considered non-atomic. The motivation is to capture certain phenomena that naturally occur around concepts with attributes, including presupposition and contraries. I present logical particulars of the logic, which is then mapped onto formal semantics. Two linguistically interesting semantics will be considered. Decidability is shown.

§1. Introduction. I present a logic that expresses gradual shifts in domain of discourse. The motivation is to capture certain peculiar phenomena about concepts/objects and other concepts/objects[1] as their attributes. The first such phenomenon is that extension of a concept alters when it becomes an attribute to other concepts. Also a concept that is specified another concept as its attribute

[1] These two terms will not be strongly distinguished in this work. An object may exist by itself, but to reason about relation between objects or just to speak about them, it is, as we presume, concepts referring to the objects that we reason/speak about.

becomes an intension [5, 22, 6] of the concept which itself is an extensional concept. Consider for instance ‘brooch’ed ‘hat’ and ‘brooch’. The ‘brooch’ in the former is an attribute to ‘hat’. By definition, if anything is an attribute to something at all, it must be found among all that can be an attribute to it. Whereas the extension of ‘brooch’ in the latter is only delimited by our understanding about ‘brooch’, that in the former as an attribute of ‘hat’ is delimited also by our understanding about the concept ‘hat’ (needless to say, only if the understanding of ours should permit ‘brooch’ as an attribute at all). But this is not all. The attribute in turn specialises the ‘hat’ to which it is an attribute: the ‘brooch’ed ‘hat’ forms an intension of the ‘hat’, and itself becomes an extensional concept “brooch’ed ‘hat”.

The shift in extension is not typically observed in formal logics, be they temporal, epistemic, modal *etc.* Some exceptions that challenge the norm are a kind of spatio-temporal logics [9, 23] and some kinds of context logics (*Cf.* [21, 14]) in the line of [4, 24]. In [9] for instance, Gabelaia *et al.* consider the definition of EU at a point of time and at another point of time. If some countries are merged into the current EU, then the term EU will remain EU at the future time reference as it is now, but the spaces that the two EU occupy are not the same. Similar phenomena are occurring in the relation between concepts/objects and their attributes. However, unlike the case of the spatio-temporal logics, there is no external and global space for them: there are only those spaces generated by the (extension of) concepts themselves. The stated (re-)action of intension/extension within an attributed concept/object is another intriguing feature that has not been formalised before.

Another point about the concept is that a concept in itself, which is to say,

an atomic concept which does not itself possess any other concepts as its attributes, is almost certainly imperceptible,[2] and hence almost certainly cannot be reasoned about. Typically, however, formal/symbolic logics assume smallest entities upon which other expressions are formed. In this work I challenge the assumption, and materialise the observation that we cannot tell apart - that is, *we cannot know* - whether a so-regarded atomic entity is atomic or is just atomic enough not to be considered non-atomic. I present a logic in which every entity is non-atomic, reflecting our intuition about the concept.[3]

Strikingly we can represent both extensional shifts and non-atomicity using the familiar classical logic only (but the results extend to other Boolean logics); with many domains of discourse.[4] The idea is as follows. We shall define a binary connective $\gtrdot$ over classical logic instances. As an example, $\mathsf{Hat} \gtrdot \mathsf{Brooch}$ reads as; Hat is, and under the presupposition that Hat is, Brooch is as its attribute. In this simple expression there are two domains of discourse: one in which Hat in $\mathsf{Hat} \gtrdot \mathsf{Brooch}$ is being discussed; and one in which Brooch in $\mathsf{Hat} \gtrdot \mathsf{Brooch}$ is being discussed. The second domain of discourse as a whole is delimited by the (extension of) Hat that gives rise to it. $\mathsf{Hat} \gtrdot \mathsf{Brooch}$ is an intension of Hat, and itself forms an extensional concept. The non-atomicity

[2] Let us arbitrarily suppose the concept hat, and let us conduct an experiment as to if we could perceive the hat in itself as something differing from nothingness for which any scenario where it comes with an attribute is inconceivable. To begin with, if the word hat evoked in our mind any specific hat with specific colour and shape, we first remove the colour out of it. If the process should make it transparent, we then remove the transparentness away from it. And if there should be still some things that are by some means perceivable as having originated from it, then because they are an attribute of the hat, we again remove any one of them. If *the humanly no longer detectable something is not nothingness* is not itself contradictory, then there must be still some quality originating in the hat that makes the something differ from nothingness. Now the question is whether the something can be perceived at all to be different from nothingness. Intuition speaks otherwise.

[3] The utility of logical non-atomicity is noted in a recent work [17] in programming community. The idea of logical non-atomicity in formal/symbolic logic itself, however, appears previously in the immediately preceding work to the current paper, namely, in [2]; as well as, and more bluntly, in its variation as a technical report.

[4] The use of multiple domains of discourse is also notable in contextual logics. Connections to those will be mentioned at the end of this work.

of concepts is captured without breaking the properties of classical logic. The ideas are that every concept has attributes, but that the attributes are not discussed in the same domain of discourse as the concept is discussed in. From within the domain of discourse discussing Hat in $\mathsf{Hat} \gg \mathsf{Brooch}$, it cannot be perceived whether it has or has not attributes, *i.e.* whether it is atomic or is just atomic enough not to be considered non-atomic.

We can also explain some reasonably common every-day linguistic phenomenon with this connective. Let us turn to an example.

Episode. There is a tiny hat shop in our town, having the following in stock:

1. 3 types of hats: orange hats, green hats ornamented with some brooch, and blue hats decorated with some white hat-shaped accessory. Only the green and the blue hats are displayed in the shop.
2. 2 types of shirts: yellow and blue shirts. Only the blue shirts are displayed in the shop.

A young man has come to the hat shop. After a while he asks the shop owner, a lady of many a year of experience in hat-making; “Have you got a yellow hat?” Knowing that it is not in her shop, she answers; “No, I do not have it in stock,” negating the possibility that there is one in stock at her shop at the present point of time. *Period*.

“What is she actually denying about?” is our question, however. It is plausible that, in delivering the answer, the question posed may have allowed her to infer that the young man was looking for a hat, a yellow hat in particular. Then the answer may be followed by she saying; “ . . . but I do have hats with different colours including ones not currently displayed.” That is, while she denies the presence of a yellow hat, she still presumes the availability of hats of which she

reckons he would like to learn. It does not appear so unrealistic to suppose such a thought of hers that he may be ready to compromise his preference for a yellow hat with some non-yellow one, possibly an orange one in stock, given its comparative closeness in hue to yellow.

Now, what if the young man turned out to be a town-famous collector of yellow articles? Then it may be that from his question she had divined instead that he was looking for something yellow, a yellow hat in particular, in which case her answer could have been a contracted form of “No, I do not have it in stock, but I do have a yellow shirt nonetheless (as you are looking after, I suppose?)”

Either way, these somewhat-appearing-to-be partial negations contrast with the classical negation with which her answer can be interpreted only as that she does not have a yellow hat, nothing less, nothing more, with no restriction in the range of possibilities outside it.

The explanation that I wish to provide is that in the first case she actually means $\mathsf{Hat} \gtrdot \neg\mathsf{Yellow}$, presuming the main concept Hat but negating Yellow as its attribute in a different domain of discourse in which its attributes can be discussed; and in the second case she actually means $\mathsf{Yellow} \gtrdot \neg\mathsf{Hat}$ with the main concept Yellow presumed but Hat denied as its attribute. Like this manner, gradual classical logic that I propose here can capture partial negation, which is known as contrariety in the pre-Fregean term logic from the Aristotle’s era [16], as well as nowadays more orthodox contradictory negation. Here we illustrated attribute negation. Complementary, we may also consider object negation of the kind $\neg\mathsf{Hat} \gtrdot \mathsf{Yellow}$, as well as more orthodox negation of the sort $\neg(\mathsf{Hat} \gtrdot \mathsf{Yellow})$ which I call attributed-object negation.

My purpose is to assume attributed concepts/objects[5] as primitive entities and

[5] In the rest I simply write attributed objects, assuming that it is clear that we do not strongly distinguish the two terms in the context of this work.

to analyse the logical behaviour of $\gtrdot$ in interaction with the other familiar connectives in classical logic. In this logic, the sense of 'truth', a very fundamental property of classical logic, gradually shifts[6] by domains of discourse moving deeper into attributes of (attributed) objects. As for inconsistency, if there is an inconsistent argument within a discourse on attributed objects, wherever it may be that it is occurring, the reasoning part of which is inconsistent cannot be said to be consistent. For this reason it remains in gradual classical logic just as strong as is in standard classical logic.

1.1. Structure of this work. Shown below is the organisation of this work. The basic conceptual core is formed in Section 1, Section 2, which is put into formal semantics in Section 3. Decidability of the logic is proved in Section 4. After the foundation is laid down, more advanced observations will be made about the object-attribute relation. They will be found in Section 5. Section 6 concludes with prospects.

- Development of gradual classical logic (Sections 1 and Section 2).
- A formal semantics of gradual classical logic and a proof that it is not para-consistent/inconsistent (Section 3).
- Decidability of gradual classical logic (Section 4).
- Advanced materials: the notion of recognition cut-offs, and an alternative presentation of gradual classical logic (Section 5).
- Conclusion (Section 6).

§2. Gradual Classical Logic: Logical Particulars. In this section we shall look into logical particulars of gradual classical logic. Some familiarity with

[6]This should not be confused with the idea of many truths in a single domain of discourse [15, 13].

propositional classical logic, in particular with how the logical connectives behave, is presumed. Mathematical transcriptions of gradual classical logic are found in the next section.

2.1. Logical connective for object/attribute and interactions with negation. We shall dedicate the symbol $\gg$ to represent the object-attribute relation. The usage of the new connective is fixed to take the form $\mathsf{Object}_1 \gg \mathsf{Object}_2$. It denotes an attributed object. Object_1 is more generic an object than $\mathsf{Object}_1 \gg \mathsf{Object}_2$ (Object_2 acting as an attribute to Object_1 makes Object_1 more specific). The schematic reading is as follows: "It is true that Object_1 is, and it is true that it has Object_2 as its attribute." Now, this really is a short-form of the following expression: "It is true by some sense of truth X reigning over the domain of discourse discussing Object_1 that Object_1 is judged existing in the domain of discourse,[7] and it is true by some sense of truth Y reigning over the domain of discourse discussing Object_2 as an attribute to Object_1 that Object_1 is judged having Object_2 as its attribute." Also, this reading is what is meant when we say that "It is true that $\mathsf{Object}_1 \gg \mathsf{Object}_2$ is," where the sense of the truth Z judging this statement has relation to X and Y, in order for compatibility. I take these side-remarks for granted in the rest without explicit stipulation. Given an attributed object $\mathsf{Object}_1 \gg \mathsf{Ojbect}_2$, $\neg(\mathsf{Object}_1 \gg \mathsf{Object}_2)$ expresses its attributed object negation, $\neg\mathsf{Object}_1 \gg \mathsf{Object}_2$ its object negation and $\mathsf{Object}_1 \gg \neg\mathsf{Object}_2$ its attribute negation. Again the schematic readings for them are, respectively;

- It is false that $\mathsf{Object}_1 \gg \mathsf{Object}_2$ is.
- It is false that Object_1 is, but it is true that some non-Object_1 is which has an attribute of Object_2.
- It is true that Object_1 is, but it is false that it has an attribute of Object_2.

[7] As must be the case, a domain of discourse defines what can be talked about, which itself does not dictate that all the elements that are found in the domain are judged existing.

The presence of negation flips "It is true that ... " into "It is false that ... " and vice versa. But it should be also noted how negation acts in attribute negations and object/attribute negations. Several specific examples constructed parodically from the items in the hat shop episode are;

1. $\text{Hat} \gtrdot \text{Yellow}$: It is true that hat is, and it is true that it has the attribute of being yellow (that is, it is yellow).
2. $\text{Yellow} \gtrdot \text{Hat}$: It is true that yellow is, and it is true that it has hat as its attribute.
3. $\text{Hat} \gtrdot \neg\text{Yellow}$: It is true that hat is, but it is false that it is yellow.
4. $\neg\text{Hat} \gtrdot \text{Yellow}$: It is false that hat is, but it is true that yellow object (which is not hat) is.
5. $\neg(\text{Hat} \gtrdot \text{Yellow})$: Either it is false that hat is, or if it is true that hat is, then it is false that it is yellow.

2.2. Object/attribute relation and conjunction. We examine specific examples first involving $\gtrdot$ and $\wedge$ (conjunction), and then observe what the readings imply.

1. $\text{Hat} \gtrdot (\text{Green} \wedge \text{Brooch})$: It is true that hat is, and it is true that it is green and brooched.
2. $(\text{Hat} \gtrdot \text{Green}) \wedge (\text{Hat} \gtrdot \text{Brooch})$: for one, it is true that hat is, and it is true that it is green; for one, it is true that hat is, and it is true that it is brooched.
3. $(\text{Hat} \wedge \text{Shirt}) \gtrdot \text{Yellow}$: It is true that hat and shirt are, and it is true that they are yellow.
4. $(\text{Hat} \gtrdot \text{Yellow}) \wedge (\text{Shirt} \gtrdot \text{Yellow})$: for one, it is true that hat is, and it is true that it is yellow; for one, it is true that shirt is, and it is true that it is yellow.

By now it has hopefully become clear that by *existential facts as truths* I do not mean how many of a given (attributed) object exist: in gradual classical logic, cardinality of objects (Cf. Linear Logic [12]) is not what it must be responsible

for, but only the facts themselves of whether any of them exist in a given domain of discourse, which is in line with classical logic.[8] Hence they univocally assume a singular rather than a plural form, as in the examples inscribed so far. The first and the second, and the third and the fourth, then equate.[9] Nevertheless, it is still important that we analyse them with a sufficient precision. In the third and the fourth where the same attribute is shared among several objects, the attribute of being yellow ascribes to all of them. Therefore those expressions are a true statement only if (1) there is an existential fact that both hat and shirt are and (2) being yellow is true for the existential fact (formed by existence of hat and that of shirt). Another example is in Figure 1.

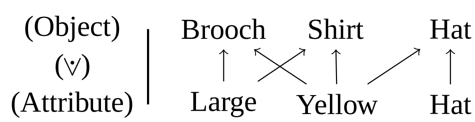

FIGURE 1. Illustration of an expression $((\text{Brooch} \wedge \text{Shirt}) \geqslant \text{Large}) \wedge ((\text{Brooch} \wedge \text{Shirt} \wedge \text{Hat}) \geqslant \text{Yellow}) \wedge (\text{Hat} \geqslant \text{Hat})$: the existential fact of the attribute large depends on the existential facts of brooch and shirt; the existential fact of the attribute of being yellow depends on the existential facts of brooch, shirt and hat; and the existential fact of the attribute hat depends on the existential fact of hat to which it is an attribute.

2.3. Object/attribute relation and disjunction. We look at examples first.

1. $\text{Hat} \geqslant (\text{Hat} \vee \text{Brooch})$: It is true that hat is, and it is true that it is either hatted or brooched.
2. $(\text{Hat} \geqslant \text{Hat}) \vee (\text{Hat} \geqslant \text{Brooch})$: At least either that it is true that hat is and it is true that it is hatted, or that it is true that hat is and it is true that it is brooched.

[8] That proposition A is true and that proposition A is true mean that proposition A is true; the subject of this sentence is equivalent to the object of its.

[9] I will also touch upon an alternative interpretation in Section 5, as an advanced material: just as there are many modal logics with a varying degree of strength of modalities, so does it seem that more than one interpretations about $\geqslant$ in interaction with the other logical connectives can be studied.

3. $(\text{Hat} \vee \text{Shirt}) \gtrdot \text{Yellow}$: It is true that at least either hat or shirt is, and it is true that whichever is existing (or both) is (or are) yellow.
4. $(\text{Hat} \gtrdot \text{Yellow}) \vee (\text{Shirt} \gtrdot \text{Yellow})$: At least either it is true that hat is and it is true that it is yellow, or it is true that shirt is and it is true that it is yellow.

Just as in the previous sub-section, here again 1) and 2), and 3) and 4) are equivalent. However, in the cases of 3) and 4), we have that the existential fact of the attribute yellow depends on that of hat or shirt, whichever is existing, or that of both if they both exist.[10]

2.4. Nestings of object/attribute relations. An expression of the kind $(\mathsf{Object}_1 \gtrdot \mathsf{Object}_2) \gtrdot \mathsf{Object}_3$ is ambiguous. But we begin by listing examples and then move onto analysis of the readings of the nesting of the relations.

1. $(\text{Hat} \gtrdot \text{Brooch}) \gtrdot \text{Green}$: It is true that hat is, and it is true that it is brooched. It is true that the object thus described is green.
2. $\text{Hat} \gtrdot (\text{Hat} \gtrdot \text{White})$: It is true that hat is, and it is true that it has the attribute of which it is true that hat is and that it is white. (More simply, it is true that hat is, and it is true that it is white-hatted.)
3. $\neg(\text{Hat} \gtrdot \text{Yellow}) \gtrdot \text{Brooch}$: Either it is false that hat is, or else it is true that hat is but it is false that it is yellow.[11] If it is false that hat is, then it is true that brooched object (which obviously cannot be hat) is. If it is true that hat is but it is false that it is yellow, then it is true that the object thus described is brooched.

Note that to say that Hat $\gtrdot$ Brooch (brooched hat) is being green, we must mean to say that the object to the attribute of being green, *i.e.* hat, is green. It is on the other hand unclear if green brooched hat should or should not mean that the brooch, an accessory to hat, is also green. But common sense about adjectives

[10]In classical logic, that proposition A or proposition B is true means that at least one of the proposition A or the proposition B is true though both can be true. Same goes here.
[11] This is the reading of $\neg(\text{Hat} \gtrdot \text{Yellow})$.

dictates that such be simply indeterminate. It is reasonable for (Hat $\gtrdot$ Brooch) $\gtrdot$ Green, while if we have (Hat $\gtrdot$ Large) $\gtrdot$ Green, ordinarily speaking it cannot be the case that the attribute of being large is green. Therefore we enforce that $(\mathsf{Object}_1 \gtrdot \mathsf{Object}_2) \gtrdot \mathsf{Object}_3$ amounts to $(\mathsf{Object}_1 \gtrdot \mathsf{Object}_3) \wedge ((\mathsf{Object}_1 \gtrdot \mathsf{Object}_2) \vee (\mathsf{Object}_1 \gtrdot (\mathsf{Object}_2 \gtrdot \mathsf{Object}_3)))$ in which disjunction as usual captures the indeterminacy. No ambiguity is posed in 2), and 3) is understood in the same way as 1).

2.5. Two nullary logical connectives. Now we examine the nullary logical connectives $\top$ and $\bot$ which denote, in classical logic, the concept of the truth and that of the inconsistency. In gradual classical logic $\top$ denotes the concept of the presence and $\bot$ denotes that of the absence. Several examples for the readings are;

1. $\top \gtrdot \text{Yellow}$: It is true that yellow object is.
2. $\text{Hat} \gtrdot (\top \gtrdot \text{Yellow})$: It is true that hat is, and it is true that it has the following attribute of which it is true that it is yellow object.
3. $\bot \gtrdot \text{Yellow}$: It is true that nothingness is, and it is true that it is yellow.
4. $\text{Hat} \gtrdot \top$: It is true that hat is.
5. $\text{Hat} \gtrdot \bot$: It is true that hat is, and it is true that it has no attributes.
6. $\bot \gtrdot \bot$: It is true that nothingness is, and it is true that it has no attributes.

It is illustrated in 1) and 2) how the sense of the 'truth' is delimited by the object to which it acts as an attribute. For the rest, however, there is a point which is not so vacuous as not to merit a consideration, and to which I in fact append the following postulate.

POSTULATE 1. *That which does not have any attribute cannot be distinguished from nothingness for which any scenario where it comes with an attribute is inconceivable. Conversely, anything that remains once all the attributes have been removed from a given object is nothingness.*

With it, the item 3) which asserts the existence of nothingness is contradictory. The item 4) then behaves as expected in that Hat which is asserted with the presence of attribute(s) is just as generic a term as Hat itself is. The item 5) which asserts the existence of an object with no attributes again contradicts Postulate 1. The item 6) illustrates that any attributed object in some part of which has turned out to be contradictory remains contradictory no matter how it is to be extended: a $\perp$ cannot negate another $\perp$. Cf. the footnote 2 for the plausibility of the postulate.

§3. Mathematical mappings: syntax and semantics. In this section a semantics of gradual classical logic is formalised. We assume in the rest of this document;

- $\mathbb{N}$ denotes the set of natural numbers including 0.
- $\wedge^\dagger$ and $\vee^\dagger$ are two binary operators on Boolean arithmetic. The following laws hold; $1 \vee^\dagger 1 = 1 \vee^\dagger 0 = 0 \vee^\dagger 1 = 1$, $0 \wedge^\dagger 0 = 0 \wedge^\dagger 1 = 1 \wedge^\dagger 0 = 0$, $1 \wedge^\dagger 1 = 1$, and $0 \vee^\dagger 0 = 0$.
- $\wedge^\dagger$, $\vee^\dagger$ $\rightarrow^\dagger$, $\neg^\dagger$, $\exists$ and $\forall$ are meta-logical connectives: conjunction, disjunction,[12] material implication, negation, existential quantification and universal quantification, whose semantics follow those of standard classical logic. We abbreviate $(A \rightarrow^\dagger B) \wedge^\dagger (B \rightarrow^\dagger A)$ by $A \leftrightarrow^\dagger B$.

[12] These two symbols are overloaded. Save whether truth values or the ternary values are supplied as arguments, however, the distinction is clear from the context in which they are used.

- Binding strength of logical or meta-logical connectives is $[\neg] \gg [\wedge \ \vee] \gg [\gtrdot] \gg [\forall \ \exists] \gg [\neg^\dagger] \gg [\wedge^\dagger \ \vee^\dagger] \gg [\rightarrow^\dagger] \gg [\leftrightarrow^\dagger]$ in the order of decreasing precedence. Those that belong to the same group are assumed having the same precedence.
- For any binary connectives $?$, for any $i, j \in \mathbb{N}$ and for $!_0, !_1, \cdots, !_j$ that are some recognisable entities, $?_{i=0}^{j} !_i$ is an abbreviation of $(!_0)?(!_1)? \cdots ?(!_j)$.
- For the unary connective $\neg$, $\neg\neg!$ for some recognisable entity ! is an abbreviation of $\neg(\neg!)$. Further, $\neg^k!$ for some $k \in \mathbb{N}$ and some recognisable entity ! is an abbreviation of $\underbrace{\neg \cdots \neg}_{k}!$.
- For the binary connective $\gtrdot$, $!_0 \gtrdot !_1 \gtrdot !_2$ for some three recognisable entities is an abbreviation of $!_0 \gtrdot (!_1 \gtrdot !_2)$.

On this preamble we shall begin.

3.1. Development of semantics. The set of literals in gradual classical logic is denoted by $\mathcal{A}$ whose elements are referred to by a with or without a sub-script. This set has a countably many number of literals. Given a literal $a \in \mathcal{A}$, its complement is denoted by a^c which is in $\mathcal{A}$. As usual, we have $\forall a \in \mathcal{A}.(a^c)^c = a$. The set $\mathcal{A} \cup \{\top\} \cup \{\bot\}$ where $\top$ and $\bot$ are the two nullary logical connectives is denoted by $\mathcal{S}$. Its elements are referred to by s with or without a sub-script. Given $s \in \mathcal{S}$, its complement is denoted by s^c which is in $\mathcal{S}$. Here we have $\top^c = \bot$ and $\bot^c = \top$. The set of formulas is denoted by $\mathfrak{F}$ whose elements, F with or without a sub-/super-script, are finitely constructed from the following grammar;

$F := s \mid F \wedge F \mid F \vee F \mid \neg F \mid F \gtrdot F$

We now develop semantics. This is done in two parts: we do not outright jump to the definition of valuation (which we could, but which we simply do not choose for succinctness of the proofs of the main results). Instead, just as we only need consider negation normal form in classical logic because every classical logic

formula definable has a reduction into a normal form, so shall we first define rules for formula reductions (for any $F_1, F_2, F_3 \in \mathfrak{F}$):

- $\forall s \in \mathcal{S}.\neg s \mapsto s^c$ ($\neg$ reduction 1).
- $\neg(F_1 \wedge F_2) \mapsto \neg F_1 \vee \neg F_2$ ($\neg$ reduction 2).
- $\neg(F_1 \vee F_2) \mapsto \neg F_1 \wedge \neg F_2$ ($\neg$ reduction 3).
- $\neg(s \gtrdot F_2) \mapsto s^c \vee (s \gtrdot \neg F_2)$ ($\neg$ reduction 4).
- $(F_1 \gtrdot F_2) \gtrdot F_3 \mapsto (F_1 \gtrdot F_3) \wedge ((F_1 \gtrdot F_2) \vee (F_1 \gtrdot F_2 \gtrdot F_3))$ ($\gtrdot$ reduction 1).
- $(F_1 \wedge F_2) \gtrdot F_3 \mapsto (F_1 \gtrdot F_3) \wedge (F_2 \gtrdot F_3)$ ($\gtrdot$ reduction 2).
- $(F_1 \vee F_2) \gtrdot F_3 \mapsto (F_1 \gtrdot F_3) \vee (F_2 \gtrdot F_3)$ ($\gtrdot$ reduction 3).
- $F_1 \gtrdot (F_2 \wedge F_3) \mapsto (F_1 \gtrdot F_2) \wedge (F_1 \gtrdot F_3)$ ($\gtrdot$ reduction 4).
- $F_1 \gtrdot (F_2 \vee F_3) \mapsto (F_1 \gtrdot F_2) \vee (F_1 \gtrdot F_3)$ ($\gtrdot$ reduction 5).

DEFINITION 1 (Domain function/valuation frame). *Let $\mathcal{S}^*$ denote the set union of (A) the set of finite sequences of elements of $\mathcal{S}$ and (B) a singleton set $\{\epsilon\}$ denoting an empty sequence. We define a domain function $D : \mathcal{S}^* \rightarrow 2^{\mathcal{S}}$. We define a valuation frame as a 2-tuple: (I, J), where $\mathsf{I} : \mathcal{S}^* \times \mathcal{S} \rightarrow \{0, 1\}$ is what we call local interpretation and $\mathsf{J} : \mathcal{S}^* \backslash \{\epsilon\} \rightarrow \{0, 1\}$ is what we call gloal interpretation. The following are defined to satisfy for all $k \in \mathbb{N}$ and for all $s_0, \ldots, s_k \in \mathcal{S}$.*

Regarding domains of discourse:

- *For all $s^* \in \mathcal{S}^*$, $D(s^*)$ is closed under complementation, and has at least $\top$ and $\bot$.*

Regarding local interpretations:

- $[\mathsf{I}(s_0.s_1.\ldots.s_{k-1}, \top) = 1]$[13] *($\mathsf{I}$ valuation of $\top$).*
- $[\mathsf{I}(s_0.s_1.\ldots.s_{k-1}, \bot) = 0]$ *(That of $\bot$).*

[13]Simply for a presentation purpose, we use a dot such as $s_1^*.s_2^*$ for $s_1^*, s_2^* \in \mathcal{S}^*$ to show that $s_1^*.s_2^*$ is an element of $\mathcal{S}^*$ in which s_1^* is the preceding constituent and s_2^* the following constituent of $s_1^*.s_2^*$. When $k = 0$, we assume that $s_0.s_1.\ldots.s_{k-1} = \epsilon$. Same applies in the rest.

- $\forall a_k \in D(s_0.s_1.\ldots.s_{k-1}).[\mathsf{I}(s_0.s_1.\ldots.s_{k-1}, a_k) = 0] \vee^\dagger$ $[\mathsf{I}(s_0.s_1.\ldots.s_{k-1}, a_k) = 1]$ *(That of a literal).*
- $\forall a_k \in D(s_0.s_1.\ldots.s_{k-1}).[\mathsf{I}(s_0.s_1.\ldots.s_{k-1}, a_k) = 0] \leftrightarrow^\dagger$ $[\mathsf{I}(s_0.s_1.\ldots.s_{k-1}, a_k^c) = 1]$ *(That of a complement).*
- $[\mathsf{I}(s_0.s_1.\ldots.s_{k-1}, s_k) = \mathsf{I}(s'_0.s'_1.\ldots.s'_{k-1}, s_k)]$ *(Synchronization condition on* I *interpretation; this reflects the dependency of the existential fact of an attribute to the existential fact of objects to which it is an attribute).*

Regarding global interpretations:

- $[\mathsf{J}(s_0.s_1.\ldots.s_k) = 1] \leftrightarrow^\dagger \forall i \in \mathbb{N}. \bigwedge_{i=0}^{\dagger k}[\mathsf{I}(s_0.s_1.\ldots.s_{i-1}, s_i) = 1]$ *(Non-contradictory* J *valuation).*
- $[\mathsf{J}(s_0.s_1.\ldots.s_k) = 0] \leftrightarrow^\dagger \exists i \in \mathbb{N}.[i \leq k] \wedge^\dagger [\mathsf{I}(s_0.s_1.\ldots.s_{i-1}, s_i) = 0]$ *(Contradictory* J *valuation).*

Note that the global interpretation is completely characterised by the local interpretation. What we will need in the end are global interpretations; local interpretations are for intermediate value calculations for the ease of presentation of the semantics and of proofs of the main results. In the rest, we assume that any literal that appears in a formula is in a domain of discourse.

DEFINITION 2 (Valuation). *Suppose a valuation frame* $\mathfrak{M} = (\mathsf{I}, \mathsf{J})$. *The following are defined to hold for all* $F_1, F_2 \in \mathfrak{F}$ *and for all* $k \in \mathbb{N}$*:*

- $[\mathfrak{M} \models s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k] = \mathsf{J}(s_0.s_1.\ldots.s_k)$.
- $[\mathfrak{M} \models F_1 \wedge F_2] = [\mathfrak{M} \models F_1] \wedge^\dagger [\mathfrak{M} \models F_2]$.
- $[\mathfrak{M} \models F_1 \vee F_2] = [\mathfrak{M} \models F_1] \vee^\dagger [\mathfrak{M} \models F_2]$.

The notions of validity and satisfiability are as usual.

DEFINITION 3 (Validity/Satisfiability). *A formula* $F \in \mathfrak{F}$ *is said to be satisfiable in a valuation frame* $\mathfrak{M}$ *iff* $1 = [\mathfrak{M} \models F]$*; it is said to be valid iff it is*

satisfiable for all the valuation frames; it is said to be invalid iff $0 = [\mathfrak{M} \models F]$ *for some valuation frame* $\mathfrak{M}$*; it is said to be unsatisfiable iff it is invalid for all the valuation frames.*

3.2. Study on the semantics. We have not yet formally verified some important points. Are there, firstly, any formulas $F \in \mathfrak{F}$ that do not reduce into some value-assignable formula? Secondly, what if both $1 = [\mathfrak{M} \models F]$ and $1 = [\mathfrak{M} \models \neg F]$, or both $0 = [\mathfrak{M} \models F]$ and $0 = [\mathfrak{M} \models \neg F]$ for some $F \in \mathfrak{F}$ under some $\mathfrak{M}$? Thirdly, should it happen that $[\mathfrak{M} \models F] = 0 = 1$ for any formula F, given a valuation frame?

If the first should hold, the semantics - the reductions and valuations as were presented in the previous sub-section - would not assign a value (values) to every member of $\mathfrak{F}$ even with the reduction rules made available. If the second should hold, we could gain $1 = [\mathfrak{M} \models F \wedge \neg F]$, which would relegate this gradual logic to a family of para-consistent logics - quite out of keeping with my intention. And the third should never hold, clearly.

Hence it must be shown that these unfavoured situations do not arise. An outline to the completion of the proofs is;

1. to establish that every formula has a reduction through $\neg$ and $\gg$ reductions into some formula F for which it holds that $\forall \mathfrak{M}.[\mathfrak{M} \models F] \in \{0, 1\}$, to settle down the first inquiry.
2. to prove that any formula F to which a value 0/1 is assignable *without the use of the reduction rules* satisfies for every valuation frame (a) that $[\mathfrak{M} \models F] \vee^{\dagger} [\mathfrak{M} \models \neg F] = 1$ and $[\mathfrak{M} \models F] \wedge^{\dagger} [\mathfrak{M} \models \neg F] = 0$; and (b) either that $0 \neq 1 = [\mathfrak{M} \models F]$ or that $1 \neq 0 = [\mathfrak{M} \models F]$, to settle down the other inquiries partially.

3. to prove that the reduction through $\neg$ reductions and $\gtrdot$ reductions on any formula $F \in \mathfrak{F}$ is normal in that, in whatever order those reduction rules are applied to F, any $F_{\textsf{reduced}}$ in the set of possible formulas it reduces into satisfies for every valuation frame either that $[\mathfrak{M} \models F_{\textsf{reduced}}] = 1$, or that $[\mathfrak{M} \models F_{\textsf{reduced}}] = 0$, for all such $F_{\textsf{reduced}}$, to conclude.

3.2.1. *Every formula is 0/1-assignable.* We state several definitions for the first objective of ours.

DEFINITION 4 (Chains/Unit chains/Unit chain expansion). *A chain is defined to be any formula $F \in \mathfrak{F}$ such that $F = F_0 \gtrdot F_1 \gtrdot \cdots \gtrdot F_{k+1}$ for $k \in \mathbb{N}$. A unit chain is defined to be a chain for which $F_i \in \mathcal{S}$ for all $0 \leq i \leq k+1$. We denote the set of unit chains by $\mathfrak{U}$. By the head of a chain $F_a \gtrdot F_b \in \mathfrak{F}$, we mean F_a; and by the tail F_b. and by the tail some formula $F_a \in \mathfrak{F}$ satisfying (1) that F_a is not in the form $F_b \gtrdot F_c$ for some $F_b, F_c \in \mathfrak{F}$ and (2) that $F = F_a \gtrdot F_d$ for some $F_d \in \mathfrak{F}$. By the tail of a chain $F \in \mathfrak{F}$, we then mean some formula $F_d \in \mathfrak{F}$ such that $F = F_a \gtrdot F_d$ for some F_a as the head of F. Given any $F \in \mathfrak{F}$, we say that F is expanded in unit chains only if any chain that occurs in F is a unit chain.*

DEFINITION 5 (Formua length). *Let us define a function as follows.*

- $\forall s \in \mathcal{S}.\textsf{f\_len}(s) = 1$.
- $\forall F_1, F_2 \in \mathfrak{F}.\textsf{f\_len}(F_1 \wedge F_2) = \textsf{f\_len}(F_1 \vee F_2) = \textsf{f\_len}(F_1 \gtrdot F_2) = \textsf{f\_len}(F_1) + \textsf{f\_len}(F_2) + 1$.
- $\forall F_1 \in \mathfrak{F}.\textsf{f\_len}(\neg F_1) = 1 + \textsf{f\_len}(F_1)$.

Then we define the length of $F \in \mathfrak{F}$ to be $\textsf{f\_len}(F)$.

DEFINITION 6 (Maximal number of $\neg$ nesting). *Let us define a function.*

- $\forall s \in \mathcal{S}.\textsf{neg\_max}(F_0) = 0$.
- $\forall F_1, F_2 \in \mathfrak{F}.\textsf{neg\_max}(F_1 \wedge F_2) = \textsf{neg\_max}(F_1 \vee F_2) = \textsf{neg\_max}(F_1 \gtrdot F_2) = \textsf{max}(\textsf{neg\_max}(F_1), \textsf{neg\_max}(F_2))$.

- $\forall F_1 \in \mathfrak{F}.\textit{neg\_max}(\neg F_1) = 1 + \textit{neg\_max}(F_1)$.

Then we define the maximal number of $\neg$ *nesting for* $F \in \mathfrak{F}$ *to be* $\textit{neg\_max}(F)$.

We now work on the main results.

LEMMA 1 (Linking principle). *Let F_1 and F_2 be two formulas in unit chain expansion. Then it holds that $F_1 \gtrdot F_2$ has a reduction into a formula in unit chain expansion.*

PROOF. Apply $\gtrdot$ reductions 2 and 3 on $F_1 \gtrdot F_2$ into a formula in which the only occurrences of the chains are $f_0 \gtrdot F_2$, $f_1 \gtrdot F_2$, ... , $f_k \gtrdot F_2$ for some $k \in \mathbb{N}$ and some $f_0, f_1, \ldots, f_k \in \mathfrak{U} \cup \mathcal{S}$. Then apply $\gtrdot$ reductions 4 and 5 to each of those chains into a formula in which the only occurrences of the chains are: $f_0 \gtrdot g_0, f_0 \gtrdot g_1, \ldots, f_0 \gtrdot g_j$, $f_1 \gtrdot g_0$, ... , $f_1 \gtrdot g_j$, ... , $f_k \gtrdot g_0$, ... , $f_k \gtrdot g_j$ for some $j \in \mathbb{N}$ and some $g_0, g_1, \ldots, g_j \in \mathfrak{U}$. To each such chain, apply $\gtrdot$ reduction 1 as long as it is applicable. This process cannot continue infinitely since any formula is finitely constructed and finitely branching by any reduction rule, and since, on the assumptions, we can apply induction on the number of elements of $\mathcal{S}$ occurring in g_x, $0 \leq x \leq j$. The straightforward inductive proof is left to readers. The result is a formula in unit chain expansion. $\dashv$

LEMMA 2 (Reduction without negation). *Any formula $F_0 \in \mathfrak{F}$ in which no $\neg$ occurs reduces into some formula in unit chain expansion.*

PROOF. By induction on the formula length. For inductive cases, consider what F_0 actually is:

1. $F_0 = F_1 \wedge F_2$ or $F_0 = F_1 \vee F_2$: Apply induction hypothesis on F_1 and F_2.
2. $F_0 = F_1 \gtrdot F_2$: Apply induction hypothesis on F_1 and F_2 to get $F_1' \gtrdot F_2'$ where F_1' and F_2' are formulas in unit chain expansion. Then apply Lemma 1.

$\dashv$

LEMMA 3 (Reduction). *Any formula $F_0 \in \mathfrak{F}$ reduces into some formula in unit chain expansion.*

PROOF. By induction on the maximal number of $\neg$ nesting, and a sub-induction on the formula length. We quote Lemma 2 for the base cases. For the inductive cases, assume that the current lemma holds true for all the formulas with $\mathsf{neg\_max}(F_0)$ of up to k. Then we conclude by showing that it still holds true for all the formulas with $\mathsf{neg\_max}(F_0)$ of $k+1$. Now, because any formula is finitely constructed, there exist sub-formulas in which occur no $\neg$. By Lemma 2, those sub-formulas have a reduction into a formula in unit chain expansion. Hence it suffices to show that those formulas $\neg F'$ with F' already in unit chain expansion reduce into a formula in unit chain expansion, upon which inductive hypothesis applies for a conclusion. Consider what F' is:

1. s: then apply $\neg$ reduction 1 on $\neg F'$ to remove the $\neg$ occurrence.
2. $F_a \wedge F_b$: apply $\neg$ reduction 2. Then apply induction hypothesis on $\neg F_a$ and $\neg F_b$.
3. $F_a \vee F_b$: apply $\neg$ reduction 3. Then apply induction hypothesis on $\neg F_a$ and $\neg F_b$.
4. $s \gg F \in \mathfrak{U}$: apply $\neg$ reduction 4. Then apply induction hypothesis on $\neg F$.

$\dashv$

LEMMA 4. *For any $F \in \mathfrak{F}$ in unit chain expansion, there exists $v \in \{0,1\}$ such that $[\mathfrak{M} \models F] = v$ for any valuation frame.*

PROOF. Since a value 0/1 is assignable to any element of $\mathcal{S} \cup \mathfrak{U}$ by Definition 2, it is (or they are if more than one in $\{0, 1\}$) assignable to $[\mathfrak{M} \models F]$. $\dashv$

Hence we obtain the desired result for the first objective.

PROPOSITION 1. *To any $F \in \mathfrak{F}$ corresponds at least one formula F_a in unit chain expansion into which F reduces. It holds for any such F_a that $[\mathfrak{M} \models F_a] \in \{0, 1\}$ for any valuation frame.*

For the next sub-section, the following observation about the negation on a unit chain comes in handy. Let us state a procedure.

DEFINITION 7 (Procedure recursiveReduce).
The procedure given below takes as an input a formula F in unit chain expansion.[14] ***Description of recursiveReduce***$(F : \mathfrak{F})$

1. *Replace $\wedge$ in F with $\vee$, and $\vee$ with $\wedge$. These two operations are simultaneous.*
2. *Replace all the non-chains $s \in \mathcal{S}$ in F simultaneously with s^c $(\in \mathcal{S})$.*
3. *For every chain F_a in F with its head $s \in \mathcal{S}$ for some s and its tail $F_{\textit{tail}}$, replace F_a with $(s^c \vee (s \gtrdot (\text{recursiveReduce}(F_{\textit{tail}}))))$.*
4. *Reduce F via $\gtrdot$ reductions in unit chain expansion.*

Then we have the following result.

PROPOSITION 2 (Reduction of negated unit chain expansion). *Let F be a formula in unit chain expansion. Then $\neg F$ reduces via the $\neg$ and $\gtrdot$ reductions into recursiveReduce(F). Moreover recursiveReduce(F) is the unique reduction of $\neg F$.*

PROOF. For the uniqueness, observe that only $\neg$ reductions and $\gtrdot$ reduction 5 are used in the reduction of $\neg F$, and that at any point during the reduction, if there occurs a sub-formula in the form $\neg F_x$, the sub-formula F_x cannot be reduced by any reduction rules. Then the proof of the uniqueness is straightforward. $\dashv$

[14]Instead of stating in lambda calculus, we aim to be more descriptive in this work for not-so-trivial a function or a procedure, using a pseudo program.

3.2.2. *Unit chain expansions form a Boolean algebra.* We make use of disjunctive normal form in this sub-section for a simplification of proofs.

DEFINITION 8 (Disjunctive/Conjunctive normal form). *A formula $F \in \mathfrak{F}$ is defined to be in disjunctive normal form only if $\exists i, j, k \in \mathbb{N}\ \exists h_0, \cdots, h_i \in \mathbb{N}$ $\exists f_{00}, \ldots, f_{kh_k} \in \mathfrak{U} \cup \mathcal{S}. F = \vee_{i=0}^{k} \wedge_{j=0}^{h_i} f_{ij}$. Dually, a formula $F \in \mathfrak{F}$ is defined to be in conjunctive normal form only if $\exists i, j, k \in \mathbb{N}\ \exists h_0, \cdots, h_i \in \mathbb{N}\ \exists f_{00}, \ldots, f_{kh_k} \in \mathfrak{U} \cup \mathcal{S}. F = \wedge_{i=0}^{k} \vee_{j=0}^{h_i} f_{ij}$.*

Now, for the second objective of ours, we prove that $\mathfrak{U} \cup \mathcal{S}$, recursiveReduce, $\vee^\dagger$ and $\wedge^\dagger$ form a Boolean algebra,[15] from which follows the required outcome.

PROPOSITION 3 (Annihilation/Identity). *For any formula F in unit chain expansion and for any valuation frame, it holds (1) that $[\mathfrak{M} \models \top \wedge F] = [\mathfrak{M} \models F]$; (2) that $[\mathfrak{M} \models \top \vee F] = [\mathfrak{M} \models \top]$; (3) that $[\mathfrak{M} \models \bot \wedge F] = [\mathfrak{M} \models \bot]$; and (4) that $[\mathfrak{M} \models \bot \vee F] = [\mathfrak{M} \models F]$.*

LEMMA 5 (Elementary complementation). *For any $s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k \in \mathfrak{U} \cup \mathcal{S}$ for some $k \in \mathbb{N}$, if for a given valuation frame it holds that $[\mathfrak{M} \models s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k] = 1$, then it also holds that $[\mathfrak{M} \models$ recursiveReduce$(s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k)] = 0$; or if it holds that $[\mathfrak{M} \models s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k] = 0$, then it holds that $[\mathfrak{M} \models$ recursiveReduce$(s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k)] = 1$. These two events are mutually exclusive.*

PROOF. For the first one, $[\mathfrak{M} \models s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k] = 1$ implies that $\mathsf{I}(\epsilon, s_0) = \mathsf{I}(s_0, s_1) = \ldots = \mathsf{I}(s_0.s_1.\ldots.s_{k-1}, s_k) = 1$. So we have; $\mathsf{I}(\epsilon, s_0^c) = \mathsf{I}(s_0, s_1^c) = \ldots = \mathsf{I}(s_0.s_1 \ldots .s_{k-1}, s_k^c)$=0 by the definition of I. Meanwhile, recursiveReduce$(s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k) = s_0^c \vee (s_0 \gtrdot ((s_1^c \vee (s_1 \gtrdot \cdots)))) = s_0^c \vee (s_0 \gtrdot s_1^c) \vee (s \gtrdot s_1 \gtrdot s_2^c) \vee \cdots \vee (s \gtrdot s_1 \gtrdot \cdots \gtrdot s_{k-1} \gtrdot s_k^c)$. Therefore $[\mathfrak{M} \models$ recursiveReduce$(s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k)] = 0 \neq 1$

[15]http://en.wikipedia.org/wiki/Boolean_algebra for the laws of Boolean algebra.

for the given valuation frame.

For the second obligation, $[\mathfrak{M} \models s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k] = 0$ implies that $[\mathsf{I}(\epsilon, s_0) = 0] \vee^\dagger [\mathsf{I}(s_0, s_1) = 0] \vee^\dagger \cdots \vee^\dagger [\mathsf{I}(s_0.s_1.\ldots.s_{k-1}, s_k) = 0]$. Again by the definition of I, we have the required result. That these two events are mutually exclusive is trivial. $\dashv$

PROPOSITION 4 (Associativity/Commutativity/Distributivity). *Given any formulas $F_1, F_2, F_3 \in \mathfrak{F}$ in unit chain expansion and any valuation frame $\mathfrak{M}$, the following hold:*

1. $[\mathfrak{M} \models F_1] \wedge^\dagger ([\mathfrak{M} \models F_2] \wedge^\dagger [\mathfrak{M} \models F_3]) = ([\mathfrak{M} \models F_1] \wedge^\dagger [\mathfrak{M} \models F_2]) \wedge^\dagger [\mathfrak{M} \models F_3]$ *(associativity 1).*
2. $[\mathfrak{M} \models F_1] \vee^\dagger ([\mathfrak{M} \models F_2] \vee^\dagger [\mathfrak{M} \models F_3]) = ([\mathfrak{M} \models F_1] \vee^\dagger [\mathfrak{M} \models F_2]) \vee^\dagger F_3$ *(associativity 2).*
3. $[\mathfrak{M} \models F_1] \wedge^\dagger [\mathfrak{M} \models F_2] = [\mathfrak{M} \models F_2] \wedge^\dagger [\mathfrak{M} \models F_1]$ *(commutativity 1).*
4. $[\mathfrak{M} \models F_1] \vee^\dagger [\mathfrak{M} \models F_2] = [\mathfrak{M} \models F_2] \vee^\dagger [\mathfrak{M} \models F_1]$ *(commutativity 2).*
5. $[\mathfrak{M} \models F_1] \wedge^\dagger ([\mathfrak{M} \models F_2] \vee^\dagger [\mathfrak{M} \models F_3]) = ([\mathfrak{M} \models F_1] \wedge^\dagger [\mathfrak{M} \models F_2]) \vee^\dagger ([\mathfrak{M} \models F_1] \wedge^\dagger [\mathfrak{M} \models F_3])$ *(distributivity 1).*
6. $[\mathfrak{M} \models F_1] \vee^\dagger ([\mathfrak{M} \models F_2] \wedge^\dagger [\mathfrak{M} \models F_3]) = ([\mathfrak{M} \models F_1] \vee^\dagger [\mathfrak{M} \models F_2]) \wedge^\dagger ([\mathfrak{M} \models F_1] \vee^\dagger [\mathfrak{M} \models F_3])$ *(distributivity 2).*

PROOF. Make use of Lemma 5 to note that each $[\mathfrak{M} \models f]$ for $f \in \mathfrak{U} \cup \mathcal{S}$ is assigned one and only one value $v \in \{0, 1\}$. Straightforward with the observation. $\dashv$

PROPOSITION 5 (Idempotence and Absorption). *Given any formula $F_1, F_2 \in \mathfrak{F}$ in unit chain expansion, for any valuation frame it holds that $[\mathfrak{M} \models F_1] \wedge^\dagger [\mathfrak{M} \models F_1] = [\mathfrak{M} \models F_1] \vee^\dagger [\mathfrak{M} \models F_1] = [\mathfrak{M} \models F_1]$ (idempotence); and that $[\mathfrak{M} \models F_1] \wedge^\dagger ([\mathfrak{M} \models F_1] \vee^\dagger [\mathfrak{M} \models F_2]) = [\mathfrak{M} \models F_1] \vee^\dagger ([\mathfrak{M} \models F_1] \wedge^\dagger [\mathfrak{M} \models F_2]) = [\mathfrak{M} \models F_1]$ (absorption).*

PROOF. Both F_1, F_2 are assigned one and only one value $v \in \{0, 1\}$. Trivial to verify. $\dashv$

We now prove the laws involving recursiveReduce.

LEMMA 6 (Elementary double negation). *Let F denote $s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k \in \mathfrak{U} \cup \mathcal{S}$ for some $k \in \mathbb{N}$. Then for any valuation frame it holds that $[\mathfrak{M} \models F] = [\mathfrak{M} \models \textsf{recursiveReduce}(\textsf{recursiveReduce}(F))]$.*

PROOF. $\textsf{recursiveReduce}(\textsf{recursiveReduce}(F)) = \textsf{recursiveReduce}(s_0^c \vee (s_0 \gtrdot s_1^c) \vee \cdots \vee (s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_{k-1} \gtrdot s_k^c)) = s_0 \wedge (s_0^c \vee (s_0 \gtrdot s_1)) \wedge (s_0^c \vee (s_0 \gtrdot s_1^c) \vee (s_0 \gtrdot s_1 \gtrdot s_2)) \wedge \cdots \wedge (s_0^c \vee (s_0 \gtrdot s_1^c) \vee \cdots \vee (s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_{k-2} \gtrdot s_{k-1}^c) \vee (s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k))$. Here, assume that the right hand side of the equation which is in conjunctive normal form is ordered, the number of terms, from left to right, strictly increasing from 1 to $k+1$. Then as the result of a transformation of the conjunctive normal form into disjunctive normal form we will have 1 (the choice from the first conjunctive clause which contains only one term s_0) $\times$ 2 (a choice from the second conjunctive clause with 2 terms s_0^c and $s_0 \gtrdot s_1$) $\times \ldots \times$ (k + 1) clauses. But almost all the clauses in $[\mathfrak{M} \models \text{(the disjunctive normal form)}]$ will be assigned 0 (trivial; the proof left to readers) so that we gain $[\mathfrak{M} \models \text{(the disjunctive normal form)}] = [\mathfrak{M} \models s_0] \wedge^\dagger [\mathfrak{M} \models s_0 \gtrdot s_1] \wedge^\dagger \cdots \wedge^\dagger [\mathfrak{M} \models s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k] = [\mathfrak{M} \models s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k]$. $\dashv$

PROPOSITION 6 (Complementation/Double negation).
For any F in unit chain expansion and for any valuation frame, we have $1 = [\mathfrak{M} \models F \vee \textsf{recursiveReduce}(F)]$ and that $0 = [\mathfrak{M} \models F \wedge \textsf{recursiveReduce}(F)]$ (complementation). Also, for any $F \in \mathfrak{F}$ in unit chain expansion and for any valuation frame we have $[\mathfrak{M} \models F] = [\mathfrak{M} \models \textsf{recursiveReduce}(\textsf{recursiveReduce}(F))]$ (double negation).

PROOF. By Proposition 4, F has a disjunctive normal form: $F = \bigvee_{i=0}^{k} \bigwedge_{j=0}^{h_i} f_{ij}$ for some $i, j, k \in \mathbb{N}$, some $h_0, \cdots, h_k \in \mathbb{N}$ and some $f_{00}, \cdots, f_{kh_k} \in \mathfrak{U} \cup \mathcal{S}$. Then we have that;

$$\mathsf{recursiveReduce}(F) = \bigwedge_{i=0}^{k} \bigvee_{j=0}^{h_i} \mathsf{recursiveReduce}(f_{ij}),$$

which, if transformed into a disjunctive normal form, will have $(h_0 + 1)$ [a choice from $\mathsf{recursiveReduce}(f_{00}), \mathsf{recursiveReduce}(f_{01}), \ldots,$ $\mathsf{recursiveReduce}(f_{0h_0})] \times (h_1 + 1)$ [a choice from $\mathsf{recursiveReduce}(f_{10}),$ $\mathsf{recursiveReduce}(f_{11}), \ldots, \mathsf{recursiveReduce}(f_{1h_1})] \times \cdots \times (h_k{+}1)$ clauses. Now if $[\mathfrak{M} \models F] = 1$, then we already have the required result. Therefore suppose that $[\mathfrak{M} \models F] = 0$. Then it holds that $\forall i \in \{0, \ldots, k\}.\exists j \in \{0, \ldots, h_i\}.([\mathfrak{M} \models f_{ij}] = 0)$. By Lemma 5, this is equivalent to saying that $\forall i \in \{0, \ldots, k\}.\exists j \in \{0, \ldots, h_i\}.([\mathfrak{M} \models \mathsf{recursiveReduce}(f_{ij})] = 1)$. But then a clause in disjunctive normal form of $[\mathfrak{M} \models \mathsf{recursiveReduce}(F)]$ exists, which is assigned 1. Dually for $0 = [\mathfrak{M} \models F \wedge \mathsf{recursiveReduce}(F)]$.

For $[\mathfrak{M} \models F] = [\mathfrak{M} \models \mathsf{recursiveReduce}(\mathsf{recursiveReduce}(F))]$, by Proposition 4, F has a disjunctive normal form: $F = \bigvee_{i=0}^{k} \bigwedge_{j=0}^{h_i} f_{ij}$ for some $i, j, k \in \mathbb{N}$, some $h_0, \ldots, h_k \in \mathbb{N}$ and some $f_{00}, \ldots, f_{kh_k} \in \mathfrak{U} \cup \mathcal{S}$. Then;

$$\mathsf{recursiveReduce}(\mathsf{recursiveReduce}(F)) = \\ \bigvee_{i=0}^{k} \bigwedge_{j=0}^{h_i} \mathsf{recursiveReduce}(\mathsf{recursiveReduce}(f_{ij})).$$

But by Lemma 6 $[\mathfrak{M} \models \mathsf{recursiveReduce}(\mathsf{recursiveReduce}(f_{ij}))] = [\mathfrak{M} \models f_{ij}]$ for each appropriate i and j. Straightforward. $\dashv$

THEOREM 1. *Denote by X the set of the expressions comprising all $[\mathfrak{M} \models f_x]$ for $f_x \in \mathfrak{U} \cup \mathcal{S}$. Then for every valuation frame, it holds that $(X, \textsf{recursiveReduce}, \wedge^{\dagger}, \vee^{\dagger})$ defines a Boolean algebra.*

PROOF. Follows from earlier propositions and lemmas. $\dashv$

3.2.3. *Gradual classical logic is neither para-consistent nor inconsistent.* To achieve the last objective we assume several notations.

DEFINITION 9 (Sub-formula notation). *Given a formula $F \in \mathfrak{F}$, we denote by $F[F_a]$ the fact that F_a occurs as a sub-formula in F. Here the definition of a sub-formula of a formula follows that which is found in standard textbooks on mathematical logic [18]. F itself is a sub-formula of F.*

DEFINITION 10 (Small step reductions). *By $F_1 \leadsto F_2$ for some formulas F_1 and F_2 we denote that F_1 reduces in one reduction step into F_2. By $F_1 \leadsto_r F_2$ we denote that the reduction holds explicitly by a reduction rule r (which is either of the 7 rules). By $F_1 \leadsto^* F_2$ we denote that F_1 reduces into F_2 in a finite number of steps including 0 step in which case F_1 is said to be irreducible. By $F_1 \leadsto^k F_2$ we denote that the reduction is in exactly k steps. By $F_1 \leadsto^*_{\{r_1, r_2, \cdots\}} F_2$ or $F_1 \leadsto^k_{\{r_1, r_2, \cdots\}} F_2$ we denote that the reduction is via those specified rules $r_1, r_2, \cdots$ only.*

DEFINITION 11 (Formula size). *Let us define a function that outputs a positive rational number, as follows. The A: B notation derives from programming practice, but simply says that A is a member of B.*

Description of** $\textsf{f\_size}(d : \mathbb{N}, l : \mathbb{N}, bool : \textsf{Boolean}, F : \mathfrak{F})$ **outputting a positive rational number

1. *If $F = s$ for some $s \in \mathcal{S}$, then return $1/4^l$.*
2. *If $F = \neg F_1$ for $F_1 \in \mathfrak{F}$, then return $(1/4^d) + \textsf{f\_size}(d, l, bool, F_1)$.*

3. *If $F = F_1 \gg F_2$, then return* f_size$(d+1, l, bool, F_1) +$ f_size$(d+1, l, bool, F_2)$.
4. *If $F = F_1 \wedge F_2$ or $F = F_1 \vee F_2$, then*
    (a) *If $bool$ is true, return max*(f_size$(d+1, l+1,$ false$, F_1)$, f_size$(d+1, l+1,$ false$, F_2)$).
    (b) *Otherwise, return max*(f_size$(d, l,$ false$, F_1)$, f_size$(d, l,$ false$, F_2)$).

Then we define the size of F to be f_size$(0, 0,$ true$, F)$.

The purpose of the last definition, including the choice of $1/4^l$, is just so that the formula size at each formula reduction does not increase. There is a one-to-one mapping between all the numbers as may be returned by this function and a subset of $\mathbb{N}$. Other than for the stated purpose, there is no rationale behind the particular decisions in the definition. Readers should not try to figure any deeper intuition, for there is none.

PROPOSITION 7 (Preliminary observation). *The following results hold.* $b \in \{$true, false$\}$.

1. f_size$(d, l, b, \neg s) \geq$ f_size(d, l, b, s^c).
2. f_size$(d, l, b, \neg(F_1 \wedge F_2)) \geq$ f_size$(d, l, b, \neg F_1 \vee \neg F_2)$.
3. f_size$(d, l, b, \neg(F_1 \vee F_2)) \geq$ f_size$(d, l, b, \neg F_1 \wedge \neg F_2)$.
4. f_size$(d, l, b, \neg(s \gg F_2)) \geq$ f_size$(d, l, b, s^c \vee (s \gg \neg F_2))$.
5. f_size$(d, l, b, (F_1 \gg F_2) \gg F_3) \geq$
   f_size$(d, l, b, (F_1 \gg F_3) \wedge ((F_1 \gg F_2) \vee (F_1 \gg F_2 \gg F_3)))$.
6. f_size$(d, l, b, F_1 \wedge F_2 \gg F_3) \geq$ f_size$(d, l, b, (F_1 \gg F_3) \wedge (F_2 \gg F_3))$.
7. f_size$(d, l, b, F_1 \vee F_2 \gg F_3) \geq$ f_size$(d, l, b, (F_1 \gg F_3) \vee (F_2 \gg F_3))$.
8. f_size$(d, l, b, F_1 \gg F_2 \wedge F_3) \geq$ f_size$(d, l, b, (F_1 \gg F_2) \wedge (F_1 \gg F_3))$.
9. f_size$(d, l, b, F_1 \gg F_2 \vee F_3) \geq$ f_size$(d, l, b, (F_1 \gg F_2) \vee (F_1 \gg F_3))$.
10. f_size$(d, l, b, F_1 \wedge F_2) =$ f_size$(d, l, b, F_2 \wedge F_1)$.
11. f_size$(d, l, b, F_1 \vee F_2) =$ f_size$(d, l, b, F_2 \vee F_1)$.

12. $\textsf{f\_size}(d, l, \textsf{b}, (F_1 \wedge F_2) \wedge F_3) = \textsf{f\_size}(d, l, \textsf{b}, F_1 \wedge (F_2 \wedge F_3))$.

13. $\textsf{f\_size}(d, l, \textsf{b}, (F_1 \vee F_2) \vee F_3) = \textsf{f\_size}(d, l, \textsf{b}, F_1 \vee (F_2 \vee F_3))$.

PROOF. Shown with an assistance of a Java program. The source code and the test cases are found in Appendix A. $\dashv$

Along with the above notations, we also enforce that $\mathcal{F}(F)$ denote the set of formulas in unit chain expansion that $F \in \mathfrak{F}$ can reduce into. A stronger result than Lemma 2 follows.

THEOREM 2 (Bisimulation). *Assumed below are pairs of formulas. F' differs from F only by the shown sub-formulas,* i.e. *F' derives from F by replacing the shown sub-formula for F' with the shown sub-formula for F and vice versa. Then for each pair (F, F') below, it holds for every valuation frame that* $[\mathfrak{M} \models$

$F_1] = [\mathfrak{M} \models F_2]$ *for all* $F_1 \in \mathcal{F}(F)$ *and for all* $F_2 \in \mathcal{F}(F')$.

$$F[F_a \wedge F_b > F_c]\ ,\ F'[(F_a > F_c) \wedge (F_b > F_c)]$$

$$F[F_a \vee F_b > F_c]\ ,\ F'[(F_a > F_c) \vee (F_b > F_c)]$$

$$F[F_a > F_b \wedge F_c]\ ,\ F'[(F_a > F_b) \wedge (F_a > F_c)]$$

$$F[F_a > F_b \vee F_c]\ ,\ F'[(F_a > F_b) \vee (F_a > F_c)]$$

$$F[(F_a > F_b) > F_c]\ ,\ F'[(F_a > F_c) \wedge ((F_a > F_b) \vee (F_a > F_b > F_c))]$$

$$F[\neg s]\ ,\ F'[s^c]$$

$$F[\neg(F_1 \wedge F_2)]\ ,\ F'[\neg F_1 \vee \neg F_2]$$

$$F[\neg(F_1 \vee F_2)]\ ,\ F'[\neg F_1 \wedge \neg F_2]$$

$$F[\neg(s > F_2)]\ ,\ F'[s^c \vee (s > \neg F_2)]$$

$$F[F_a \vee F_a]\ ,\ F'[F_a]$$

$$F[F_a \wedge F_a]\ ,\ F'[F_a]$$

$$F[F_a \wedge F_b]\ ,\ F'[F_b \wedge F_a]$$

$$F[F_a \vee F_b]\ ,\ F'[F_b \vee F_a]$$

$$F[F_a \wedge (F_b \wedge F_c)]\ ,\ F'[(F_a \wedge F_b) \wedge F_c]$$

$$F[F_a \vee (F_b \vee F_c)]\ ,\ F'[(F_a \vee F_b) \vee F_c]$$

PROOF. By simultaneous induction on the size of the formula that is not a strict sub-formula of any other formulas[16], a sub-induction on the inverse of (the number of occurrences of $\neg$ + 1)[17] and a sub-sub-induction on the inverse of (the number of occurrences of $>$ + 1). None of these are generally an intger; but there is a mapping into $\mathbb{N}$, so that a larger number maps into a larger natural number. The composite induction measure strictly decreases at each reduction

[16]That is, if $F \rightsquigarrow F_a \rightsquigarrow F_b \rightsquigarrow \ldots$, then we get $\mathsf{f\_size}(0, 0, \mathsf{true}, F), \mathsf{f\_size}(0, 0, \mathsf{true}, F_a), \mathsf{f\_size}(0, 0, \mathsf{true}, F_b)$...

[17]If $\neg$ occurs once, then we get $1/2$. If it occurs twice, then we get $1/3$.

(Cf. Appendix A). We first establish that $\mathcal{F}(F_1) = \mathcal{F}(F_2)$ (by bisimulation). One way to show that to each reduction on F' corresponds reduction(s) on F is straightforward, for we can choose to reduce F into F', thereafter synchronizing both of the reductions. Into the other way to show that to each reduction on F corresponds reduction(s) on F';

1. The first pair.
   (a) If a reduction takes place on a sub-formula which neither is a sub-formula of the shown sub-formula nor takes as its sub-formula the shown sub-formula, then we reduce the same sub-formula in F'. Induction hypothesis on the pair of the reduced formulas. (The formula size of the stated formulas is that of F in this direction of the proof).
   (b) If it takes place on a sub-formula of F_a or F_b then we reduce the same sub-formula of F_a or F_b in F'. Induction hypothesis.
   (c) If it takes place on a sub-formula of F_c then we reduce the same sub-formula of both occurrences of F_c in F'. Induction hypothesis.
   (d) If $\gtrdot$ reduction 2 takes place on F such that we have; $F[(F_a \wedge F_b) \gtrdot F_c] \rightsquigarrow F_x[(F_a \gtrdot F_c) \wedge (F_b \gtrdot F_c)]$ where F and F_x differ only by the shown sub-formulas,[18] then do nothing on F'. And $F_x = F'$. Vacuous thereafter.
   (e) If a reduction takes place on a sub-formula F_p of F in which the shown sub-formula of F occurs as a strict sub-formula ($F[(F_a \wedge F_b) \gtrdot F_c] = F[F_p[(F_a \wedge F_b) \gtrdot F_c]]$), then we have $F[F_p[(F_a \wedge F_b) \gtrdot F_c]] \rightsquigarrow F_x[F_q[(F_a \wedge F_b) \gtrdot F_c]]$. But we have $F' = F'[F'_p[(F_a \gtrdot F_c) \wedge (F_b \gtrdot F_c)]]$. Therefore we apply the same reduction on F'_p to gain; $F'[F'_p[(F_a \gtrdot F_c) \wedge (F_b \gtrdot F_c)]] \rightsquigarrow F'_x[F'_q[(F_a \gtrdot F_c) \wedge (F_b \gtrdot F_c)]]$. Induction hypothesis.
2. The second pair: Similar.

[18]This note 'where . . . ' is assumed in the remaining.

3. The third pair: Similar, except when $\neg$ reduction 4 applies such that we have; $F[\neg(s \gtrdot F_b \wedge F_c)] \rightsquigarrow F_p[s^c \vee (s \gtrdot \neg(F_b \wedge F_c))]$. By the simultaneous induction and by Proposition 7, it does not cost generality if we replace it with $F_q[s^c \vee (s \gtrdot \neg F_b \vee \neg F_c)]$ that differs from F_p only by the shown sub-formulas, which we then replace with $F_r[(s^c \vee s^c) \vee ((s \gtrdot \neg F_b) \vee (s \gtrdot \neg F_c))]$. Since $\mathsf{f\_size}(0, 0, \mathsf{true}, F_r) < \mathsf{f\_size}(0, 0, \mathsf{true}, F)$, we again replace it with $F_u[s^c \vee (s^c \vee ((s \gtrdot \neg F_b) \vee (s \gtrdot \neg F_c)))]$, and so on and so forth, to eventually arrive at $F_v[(s^c \vee (s \gtrdot \neg F_b)) \vee (s^c \vee (s \gtrdot \neg F_c))]$, without loss of generality. Meanwhile, we can reduce F' as follows. $F'[\neg((s \gtrdot F_b) \wedge (s \gtrdot F_c))] \rightsquigarrow F'_x[\neg(s \gtrdot F_b) \vee \neg(s \gtrdot F_c)] \rightsquigarrow F'_y[(s^c \vee (s \gtrdot \neg F_b)) \vee (s^c \vee (s \gtrdot \neg F_c))]$. Induction hypothesis. The other cases are straightforward.
4. The fourth pair: Similar.
5. The fifth pair:
   (a) If a reduction takes place on a sub-formula which neither is a sub-formula of the shown sub-formula nor takes as its sub-formula the shown sub-formula, then we reduce the same sub-formula in F'. Induction hypothesis.
   (b) If it takes place on a sub-formula of F_a, F_b or F_c, then we reduce the same sub-formula of all the occurrences of the shown F_a, F_b or F_c in F'. Induction hypothesis.
   (c) If $\gtrdot$ reduction 4 takes place on F such that we have; $F[(F_a \gtrdot F_b) \gtrdot F_c] \rightsquigarrow F_x[(F_a \gtrdot F_c) \wedge ((F_a \gtrdot F_b) \vee (F_a \gtrdot F_b \gtrdot F_c))]$, then do nothing on F'. And $F_x = F'$. Vacuous thereafter.
   (d) If a reduction takes place on a sub-formula F_p of F in which the shown sub-formula of F occurs as a strict sub-formula, then similar to the case 1) e).
6. The sixth pair: Straightforward.

7. The seventh and the eighth pairs: Similar.
8. The ninth pair: Similar except when either $\gtrdot$ reduction 4 or 5 takes place, which we have already covered (for the third pair).
9. The 10th pair: Mostly straightforward. Suppose $\gtrdot$ reduction 2 applies such that we have; $F[F_a \wedge F_a \gtrdot F_b] \rightsquigarrow F_p[(F_a \gtrdot F_b) \wedge (F_a \gtrdot F_b)]$, then because we have $F'[F_a \gtrdot F_b]$, we apply induction hypothesis for a conclusion. Or, suppose that a reduction takes place on a sub-formula of an occurrence of F_a such that we have; $F \rightsquigarrow F_x[F_u \wedge F_a \gtrdot F_b]$, then by the simultaneous induction, it does not cost generality if we replace it with $F_y[F_u \wedge F_u \gtrdot F_b]$ that differs from F_x only by the shown sub-formulas. Meanwhile, we apply the same reduction rule on the occurrence of F_a in F' such that we have; $F' \rightsquigarrow F'_y[F_u \gtrdot F_b]$. Induction hypothesis. Likewise for the others.
10. The 11th pairs: Similar.
11. The 12th and the 13th pairs: Straightforward.
12. The 14th and the 15th pairs: Cf. the approach for the third pair.

By the result of the above bisimulation, we now have $\mathcal{F}(F) = \mathcal{F}(F')$. However, it takes only those 5 $\gtrdot$ reductions and 4 $\neg$ reductions to derive a formula in unit chain expansion; hence we in fact have $\mathcal{F}(F) = \mathcal{F}(F_x)$ for some formula F_x in unit chain expansion. But then by Theorem 1, there could be only one value out of $\{0, 1\}$ assigned to $[\mathfrak{M} \models F_x]$, as required. $\dashv$

COROLLARY 1 (Normalisation). *Given a formula $F \in \mathfrak{F}$, denote the set of formulas in unit chain expansion that it can reduce into by $\mathcal{F}_1$. Then it holds for every valuation frame either that $[\mathfrak{M} \models F_a] = 1$ for all $F_a \in \mathcal{F}_1$ or else that $[\mathfrak{M} \models F_a] = 0$ for all $F_a \in \mathcal{F}_1$.*

By Theorem 1 and Corollary 1, we may define implication: $F_1 \supset F_2$ to be an abbreviation of $\neg F_1 \vee F_2$ - *exactly the same* - as in classical logic.

§4. Decidability. We show a decision procedure $\oint$ for universal validity of some input formula F. Also assume a terminology of 'object level', which is defined inductively. Given F in unit chain expansion, (A) if $s \in \mathcal{S}$ in F occurs as a non-chain or as a head of a unit chain, then it is said to be at the 0-th object level. (B) if it occurs in a unit chain as $s_0 \gtrdot \cdots \gtrdot s_k \gtrdot s$ or as $s_0 \gtrdot \cdots \gtrdot s_k \gtrdot s \gtrdot ...$ for some $k \in \mathbb{N}$ and some $s_0, \ldots, s_k \in \mathcal{S}$, then it is said to be at the (k+1)-th object level. Further, assume a function $\textsf{toSeq} : \mathbb{N} \to \mathcal{S}^*$ satisfying $\textsf{toSeq}(0) = \epsilon$ and $\textsf{toSeq}(k+1) = \underbrace{\top.\ldots.\top}_{k+1}$.

$\oint(F : \mathfrak{F},$ **object_level** $: \mathbb{N})$**: returning either 0 or 1**

$\backslash\backslash$ This pseudo-textsf uses $n, o : \mathbb{N}$, $F_a, F_b : \mathfrak{F}$.

L0: Duplicate F and assign the copy to F_a. If F_a is not already in unit chain expansion, then reduce it into a formula in unit chain expansion.

L1: $F_b := \textsf{EXTRACT}(F_a, \textsf{object\_level})$.

L2: $n := \textsf{COUNT\_DISTINCT}(F_b)$.

L3$_0$**:** For each $\textsf{I} : \textsf{toSeq}(\textsf{object\_level}) \times \mathcal{S}$ distinct for the n elements of $\mathcal{S}$ at the given object level, Do:

L3$_1$**:** If $\textsf{UNSAT}(F_b, \textsf{I})$, then go to **L5**.

L3$_2$**:** Else if no unit chains occur in F_a, go to **L3**$_5$.

L3$_3$**:** $o := \oint(\textsf{REWRITE}(F_a, \textsf{I}, \textsf{object\_level}), \textsf{object\_level} + 1)$.

L3$_4$**:** If $o = 0$, go to **L5**.

L3$_5$**:** End of For Loop.

L4: return 1. $\backslash\backslash$ Yes.

L5: return 0. $\backslash\backslash$ No.

EXTRACT($F : \mathfrak{F},$ **object_level** $: \mathbb{N}$**) returning** $F' : \mathfrak{F}$**:**

L0: $F' := F$.

L1: For every $s_0 \gtrdot s_1 \gtrdot \cdots \gtrdot s_k$ for some $k \in \mathbb{N}$ greater than or equal to $\textsf{object\_level}$ and some $s_0, s_1, \ldots, s_k \in \mathcal{S}$ occurring in F', replace it with $s_0 \gtrdot \cdots \gtrdot s_{\textsf{object\_level}}$.

L2: return F'.

COUNT_DISTINCT($F : \mathfrak{F}$) **returning** $n : \mathbb{N}$**:**

L0: return $n :=$ (number of distinct members of $\mathcal{A}$ in F).

UNSAT($F : \mathfrak{F}, \mathsf{I} : \mathsf{I}$**) returning true or false:**

L0: return $\textsf{true}$ if, for the given interpretation I, $[(\mathsf{I}, \mathsf{J}) \models F] = 0$. Otherwise, return $\textsf{false}$.

REWRITE($F : \mathfrak{F}, \mathsf{I} : \mathsf{I}$, **object_level** $: \mathbb{N}$**) returning** $F' : \mathfrak{F}$**:**

L0: $F' := F$.

L1: remove all the non-unit-chains and unit chains shorter than or equal to $\textsf{object\_level}$ from F'. The removal is in the following sense: if $f_x \wedge F_x$, $F_x \wedge f_x$, $f_x \vee F_x$ or $F_x \vee f_x$ occurs as a sub-formula in F' for f_x those just specified, then replace them not simultaneously but one at a time to F_x until no more reductions are possible.

L2$_0$: For each unit chain f in F', Do:

L2$_1$: if the head of f is 0 under I, then remove the unit chain from F'; else replace the head of f with $\top$.

L2$_2$: End of For Loop.

L3: return F'.

The intuition of the procedure is found within the proof below.

PROPOSITION 8 (Decidability of gradual classical logic). *Complexity of* $\oint(F, 0)$ *is at most EXPTIME.*

PROOF. We show that it is a decision procedure. That the complexity bound cannot be worse than EXPTIME is clear from the semantics (for **L0**) and from

the procedure itself. Consider **L0** of the main procedure. This reduces a given formula into a formula in unit chain expansion. In **L1** of the main procedure, we get a snapshot of the input formula. We extract from it components of the 0-th object level, and check if it is (un)satisfiable. The motivation for this operation is as follows: if the input formula is contradictory at the 0th-object level, the input formula is contradictory by the definition of J. Since we are considering validity of a formula, we need to check all the possible valuation frames. The number is determined by distinct $\mathcal{A}$ elements. **L2** gets the number (n). The For loop starting at $\mathbf{L3}_0$ iterates through the 2^n distinct interpretations. If the snapshot is unsatisfiable for any such valuation frame, it cannot be valid, which in turn implies that the input formula cannot be valid ($\mathbf{L3}_1$). If the snapshot is satisfiable and if the maximum object-level in the input formula is the 0th, *i.e.* the snapshot is the input formula, then the input formula is satisfiable for this particular valuation frame, and so we check the remaining valuation frames ($\mathbf{L3}_2$). Otherwise, if it is satisfiable and if the maximum object-level in the input formula is not the 0th, then we need to check that snapshots in all the other object-levels of the input formula are satisfiable by all the valuation frames. We do this check by recursion ($\mathbf{L3}_3$). Notice the first parameter $\mathsf{REWRITE}(F_a, \mathsf{I}, \mathsf{object\_level})$ here. This returns some formula F'. At the beginning of the sub-procedure, F' is a duplicated copy of F_a (not F_b). Now, under the particular 0-th object level interpretation I, some unit chain in F_a may be already evaluated to 0. Then we do not need consider them at any deeper object-level. So we remove them from F'. Otherwise, in all the remaining unit chains, the 0-th object gets local interpretation of 1. So we replace the $\mathcal{S}$ element at the 0-th object level with $\top$ which always gets 1.[19] Finally, all the non-chain $\mathcal{S}$ constituents and all the chains shorter than or equal to $\mathsf{object\_level}$ in F_a are irrelevant at a higher object-level.

[19]Such replacement does not preserve equivalence; equisatisfiability is preserved, however.

So we also remove them (from F'). We pass this F' and an incremented object_level to the main procedure for the recursion.

The recursive process continues either until a sub-formula passed to the main procedure turns out to be invalid, in which case the recursive call returns 0 (**L2**$_2$ and **L4** in the main procedure) to the caller who assigns 0 to o (**L2**$_4$) and again returns 0, and so on until the first recursive caller. The caller receives 0 once again to conclude that F is invalid, as expected. Otherwise, we have that F is valid, for we considered all the valuation frames. The number of recursive calls cannot be infinite. $\dashv$

§5. Analysis. In this section I will present an advanced observation about the principle of gradual logic. I will also highlight an alternative interpretation of the object-attribute relation, which is hoped to cement the idea of gradual logic. A moderate comprehension of the gists of Section 1, Section 2 and Section 3 is a pre-requisite for the first sub-section. The sub-section 5.2 assumes a full understanding of Section 3.

5.1. The notion of recognition cut-off. There are many that can be seen in the object-attribute relation. According to Postulate 1, all the (attributed) objects, so long as they remain recognisable as an object, have an extension (recall the relation between Hat and Hat $\gg \top$), *i.e.* they are not atomic; and because an attribute is also an object, the implication is that no matter how deep the ladder of attributed objects formed in a sequence of $\gg$ goes, there is no possibility that we arrive at the most precise description of the object. Observed from the other side, it means that we can always refine any given (attributed) object with more attributes as we notice ambiguities in them. For illustration, if we have Hat, then Hat $\gg$ Brooch, Hat $\gg$ Brooch $\gg$ Green, $\ldots$, are, provided that they are not contradictory, guaranteed to be a specific instance of Hat. At the same time, none of

them is a fully explicated atomic object, since every one of them has extension. Descriptions in gradual classical logic reflect our intuition about concepts (Cf. Section 1) in this manner.

One issue that must be touched upon, however, is that the arbitrary ambiguity has been the reason why natural languages are generally considered unsuited (Cf. [8, 27]) for a rigorous treatment of concepts. As can be inferred from the analysis in Section 1 but also found already in Transcendental Logic, the truth in formal logic can be only this thing or that thing that we define as the truth. But, then, the arbitrary ambiguity in entities do appear to encumber the construction of the definition of this and that things, for we seem to be hitting upon an impossibility of knowing what they are. It is then reasonable to presuppose indivisible entities that act as the building stones of the truth, which is the measure typically taken in formal/symbolic logic.

A rather different perspective about the ambiguity and atomicity of an entity was taken in this work, which is embodied in the following reasoning. - If the arbitrary ambiguity in natural descriptions must be felt universally, it would not be possible for us to coherently speak on a topic (which naturally concerns concepts referring to objects), for whatever that comes into our mind cannot be fully disambiguated. As a matter of fact, however, we have little issue in drawing a comprehensible conclusion, be it agreeable to us or not, from a discussion with our acquaintances. Concerning it, it appears that what is at stake is not the coaxing paradox: although nothing should be comprehensible, we have nonetheless comprehended the quintessence of something; but, rather, our recognition of a divisible entity as an indivisible one. As one descriptive - albeit rather impromptu - example, suppose we have an apparatus, which has almost no utility, save that it could tell if a book is in 210 mm $\times$ 297 mm. The actual judgement mechanism of the apparatus is concealed from the eyes of the users. Now, let

us say that we have applied it to several books, as the result of which some of them have turned out to be in 210 mm $\times$ 297 mm. But then it could happen that a book is in 209.993 mm $\times$ 297.001 mm, another in 210.01 mm $\times$ 297.002 mm, if we are to measure the size more accurately by another method. But the point is that, so long as the measure of judging the size is as facilitated by the apparatus, we see a given book either in 210 mm $\times$ 297 mm, or otherwise. We will not know of the variance in 210 mm $\times$ 297 mm unless the measure itself is changed. Here, a measure defining a threshold and adopting which we become indifferent to all the remaining details outside it, is what may be called a recognition cut-off, which in the above example was enforced by the apparatus.

This principle of the recognition cut-off is prominently applied in concept-manoeuvring in general, where the apparatus is nothing but a state of our mind as conditioned such as by knowledge and pre-suppositions. We understand things, and the things are understood. Yet, according to Postulate 1, it looks that they cannot be understood. Generally, this does not indicate a misuse on our part of the term: to understand. What it does indicate, on the contrary, is the susceptibility of the existence of the things to our perception and cognition which define recognition cut-offs, by the merit of which, if for instance 'I understand things' is given, it finds a cogent interpretation that 'I' referring to I under a recognition cut-off (Cf. Transcendental Logic also for Kant's observation about 'I') understands 'things' referring to the things under a recognition cut-off, so that it becomes indeed possible for one to understand things without him/her, in supporting the very possibility, being forced to accede that he/she is erring.

With the notion of the recognition cut-off, we can at last give a satisfactory justice to the subject of the arbitrary ambiguity and atomicity in gradual classical logic. In the expression $\textsf{Hat} \gg \textsf{Brooch} \gg \textsf{Green} \gg \textsf{Lamination}$, with the intended reading of the existence of "hat ornamented with a brooch in laminated green",

Hat, at least some form of it, must exist before anything, for it would be absurd to state that it be possible to reason about the attributes in the absence of an entity to which they are allegedly an attribute. At the moment of the judgement of the existence, the judgement measure cannot favour one specific hat to other hats, so long as it is hat, which therefore exerts its influence only over what is found in the specific domain of discourse in which Hat is found. Even if Hat is non-atomic, it is still judged as if atomically under the judgement setting forth a recognition cut-off. It is only when deeper attributes are cogitated that it comes to light that it was not atomic. Meanwhile the attributes involving Brooch are again judged as if atomically under the judgement measure reigning over the domain of discourse conditioned (at least) by the existence of Hat. In this manner, gradual classical logic materialises the observation that we cannot tell apart whether a so-regarded atomic entity is atomic or is just atomic enough not to be considered non-atomic.

5.1.1. *Homonyms under recognition cut-off.* More often than not, literature stresses that there is something particularly interesting about homonyms since, unless a sufficient context is given, what they denote cannot be determined. By virtue of the recognition cut-off, however, they are almost as ambiguous a description as any other common descriptions, since a description involving concepts referring to objects, according to Postulate 1, possesses the same degree of ambiguity as a homonym does - the same degree insofar as they are arbitrarily ambiguous. A 'book' identifies that what the concept points to shall be a book, but nothing more can be asserted. But this then allows us to infer that 'bow' with little to no contexts provided still identifies that what the concept refers to shall be bow. The following criticism is amply expected at this point: such an answer, for the reason that it by no means addresses the fundamental problem

that 'bow' with no context does not determine which (definition of) 'bow' it is, is absurd. However, if it were absurd, then in order to avoid the same absurdity it must be explicated what bow, supposing that enough context has been given to identify it as a violin bow, it is. An answer would burgeon the criticism of the same kind, and we would never get out of the cycle. If we are permitted to fluctuate the point of the recognition cut-off freely, it holds that a word with a context no more determines what it is than that with no context does. One description concerning concept(s) referring to objects is only comparatively less ambiguous than others.

5.2. Another interpretations of the object-attribute relation. Semantic interpretation of $\gtrdot$ is not restricted to the one that we saw in Section 2, which was formalised in Section 3. Just like in modal logic, there are other interpretations that could have a linguistic meaningfulness. In one variant, we may remove the synchronization condition on I interpretation. The motivation is that, suppose $X \gtrdot Y$, it may be that we like to say that the attribute Y varies according to what it is an attribute to: X in this case. Then, if we have $\mathsf{Hat} \gtrdot \mathsf{Green}$ and $\mathsf{Brooch} \gtrdot \mathsf{Green}$, we do not know if the same greenness is talked about for Hat and Brooch. In such an interpretation, we do not have the following distributivity: $\mathsf{Hat} \wedge \mathsf{Brooch} \gtrdot \mathsf{Green} \mapsto (\mathsf{Hat} \gtrdot \mathsf{Green}) \wedge (\mathsf{Brooch} \gtrdot \mathsf{Green})$. Another distributivity of the sort: $(\mathsf{Hat} \vee \mathsf{Brooch}) \gtrdot \mathsf{Green} \mapsto (\mathsf{Hat} \gtrdot \mathsf{Green}) \vee (\mathsf{Brooch} \gtrdot \mathsf{Green})$ would also need altered to: $(\mathsf{Hat} \gtrdot \mathsf{Green}) \vee (\mathsf{Brooch} \gtrdot \mathsf{Green}) \vee (\mathsf{Hat} \wedge \mathsf{Brooch} \gtrdot \mathsf{Green})$, covering each possibility of the existence of the objects. On the other hand, we may or may not have the rule of the sort: $(\mathsf{Hat} \gtrdot \mathsf{Green}) \gtrdot \mathsf{Brooch} \mapsto (\mathsf{Hat} \gtrdot \mathsf{Brooch}) \wedge ((\mathsf{Hat} \gtrdot \mathsf{Green}) \vee (\mathsf{Hat} \gtrdot \mathsf{Green} \gtrdot \mathsf{Brooch}))$. In our demonstration, we choose not to include this rule for brevity. We omit $\top$ and $\bot$, for $\top$ does not

behave well in $(\top \gtrdot F_1) \wedge (\top \gtrdot F_1)$ under the specified interpretation.[20] Let us formalise this logic, beginning with peripheral definitions.

DEFINITION 12 (Unit graph chain/unit graph expansion). *Given any F (no occurrences of $\top$ and $\bot$), we say that F is a unit graph chain if and only if it is recognised in the following rules.*

- *A unit chain is a unit graph chain.*
- *If F_1 is a unit graph chain and a is a literal, then $F_1 \gtrdot a$ is a unit graph chain.*
- *If a is a literal, and F_1 and F_2 are either a literal or a unit graph chain, then $F_1 \wedge F_2 \gtrdot a$ is a unit graph chain.*

We say that a given formula is in unit graph expansion if and only if all the chains that occur in the formula are a unit graph chain.

By G with or without a sub-/super-script we denote a formula that is either a literal or a unit graph chain.

The semantics is as follows. Assume that $\mathcal{I}(k)$ for $k \in \mathbb{N}$ is the power set of $\{0, 1, \ldots, k\}$ minus the empty set.

- $\forall a \in \mathcal{A}.\neg a \mapsto a^c$ ($\neg$ reduction 1).
- $\neg(F_1 \wedge F_2) \mapsto \neg F_1 \vee \neg F_2$ ($\neg$ reduction 2).
- $\neg(F_1 \vee F_2) \mapsto \neg F_1 \wedge \neg F_2$ ($\neg$ reduction 3).
- $\neg(G_0 \wedge \cdots \wedge G_k \gtrdot F_2) \mapsto \neg G_0 \vee \cdots \vee \neg G_k \vee (G_0 \wedge \cdots \wedge G_k \gtrdot \neg F_2)$ ($\neg$ reduction 4).
- $G_0 \vee G_1 \vee \cdots \vee G_{k+1} \gtrdot F \mapsto \bigvee_{I \in \mathcal{I}(k+1)}(\bigwedge_{j \in I} G_j \gtrdot F)$ ($\gtrdot$ reduction 3).[21]
- $F_1 \gtrdot F_2 \wedge F_3 \mapsto (F_1 \gtrdot F_2) \wedge (F_1 \gtrdot F_3)$ ($\gtrdot$ reduction 4).
- $F_1 \gtrdot F_2 \vee F_3 \mapsto (F_1 \gtrdot F_2) \vee (F_1 \gtrdot F_3)$ ($\gtrdot$ reduction 5).

[20] It is also recommendable that the number of elements of each domain of discourse be at least countably infinite in this interpretation.
[21] This should not be confused with $\bigvee_{I \in \mathcal{I}(k+1)}(\bigwedge_{j \in I}(G_j \gtrdot F))$.

- $F_1 \wedge (F_2 \vee F_3) \gtrdot F_4 \mapsto (F_1 \wedge F_2) \vee (F_1 \wedge F_3) \gtrdot F_4$ (obj distribution 1).
- $(F_1 \vee F_2) \wedge F_3 \gtrdot F_4 \mapsto (F_1 \wedge F_3) \vee (F_2 \wedge F_3) \gtrdot F_4$ (obj distribution 2).

We assume that the $\gtrdot$ reduction 3 applies to any $\vee$-connected unit graph chains with no regard to a particular association among the unit graph chains: it applies just as likely to $(G_0 \vee G_1) \vee (G_2 \vee G_3) \gtrdot F$ as to $G_0 \vee (G_2 \vee (G_3 \vee G_1)) \gtrdot F$. Similarly for the $\wedge$-connected formulas in $\neg$ reduction 4.

PROPOSITION 9 (Reduction of induction measure). *Let induction measure be the formula size (the main induction), the inverse of (the number of $\neg + 1$) (a sub-induction), the inverse of (the number of $\gtrdot + 1$) (a sub-sub-induction), and the inverse of (the number of $\wedge + 1$) (a sub-sub-sub-induction). Then the induction measure strictly decreases at each reduction on a given formula. Additionally, associativity and commutativity of $\wedge$ and $\vee$ do not alter the induction measure.*

PROOF. Checked with a Java program, whose source code is as found in Appendix A. The test cases are found in Appendix B. $\dashv$

DEFINITION 13 (Domains and valuations). *Let $\mathcal{T}$ denote a non-empty set that has all the elements that match the following inductive rules.*

- $\langle a \rangle$ *for* $a \in \mathcal{A}$ *is an element of* $\mathcal{T}$.
- $\lfloor a \rfloor$ *for* $a \in \mathcal{A}$ *is an element of* $\mathcal{T}$.
- *if* t *is an element of* $\mathcal{T}$, *then so are both* $\langle t \rangle$ *and* $\lfloor t \rfloor$.
- *if* t_1, t_2 *are elements of* $\mathcal{T}$, *then so are* $\langle t_1, t_2 \rangle$ *and* $\lfloor t_1 \rfloor . \lfloor t_2 \rfloor$.

We assume that $\langle\ \rangle$ *defines an unordered set:* $\langle t_1, t_2 \rangle = \langle t_2, t_1 \rangle$; $\langle t, t \rangle = \langle t \rangle$. *We also assume the following congruence relations among the elements:* $\langle \langle t \rangle \rangle \doteq \langle t \rangle$, $\langle t_1, t_2 \rangle \doteq \langle t_2, t_1 \rangle$ *and* $\lfloor \lfloor t \rfloor \rfloor \doteq \lfloor t \rfloor$. *Then by* $\dot{\mathcal{T}}$ *we denote a sub-set of* $\mathcal{T}$ *which contains only the least elements in each congruence class.*[22] *Now, let* $\mathcal{T}^*$ *denote*

[22] Here, an element in one congruence class is smaller than another if it contains a fewer number of symbols.

the set of all the finite sequences of elements of $\dot{\mathcal{T}}$*, e.g.* $t_0.t_1.\ldots.t_k$ *for some* $k \in \mathbb{N}$*, plus an empty sequence which we denote by* $\{\epsilon\}$*. An element of* $\mathcal{T}^*$ *is referred to by* t^* *with or without a sub-script. Then, we define a domain function* $D : \mathcal{T}^* \rightarrow 2^{\mathcal{A}}\backslash\emptyset$*, and a valuation frame as a 2-tuple:* (I, J)*, where* $\mathsf{I} : \mathcal{T}^* \times \mathcal{A} \rightarrow \{0, 1\}$ *is what we call local interpretation and* $\mathsf{J} : \mathcal{T}^*\backslash\{\epsilon\} \rightarrow \{0, 1\}$ *is what we call global interpretation. The following are defined to satisfy for all* $k \in \mathbb{N}$*, for all* $t^* \in \mathcal{T}^*$*, and for all* $t_0, \ldots, t_k \in \dot{\mathcal{T}}$*.*

Regarding domains of discourse:

- *For all* $t^* \in \mathcal{T}^*$*,* $D(t^*)$ *is closed under complementation and is non-empty.*

Regarding local interpretations:

- $\forall a \in D(t^*).[\mathsf{I}(t^*, a) = 0] \vee^\dagger [\mathsf{I}(t^*, a) = 1]$.
- $\forall a \in D(t^*).[\mathsf{I}(t^*, a) = 0] \leftrightarrow^\dagger [\mathsf{I}(t^*, a^c) = 1]$.

Regarding global interpretations:

- $\mathsf{J}(\lfloor t_0 \rfloor.\lfloor t_1 \rfloor.\ldots.\lfloor t_k \rfloor) = \bigwedge_{i=0}^{k} \mathsf{J}(\lfloor t_0 \rfloor.\ldots.\lfloor t_{i-1} \rfloor.t_i)$.
- $\mathsf{J}(t^*.\langle t_0, t_1, \ldots, t_k \rangle) = \bigwedge_{i=0}^{k} \mathsf{J}(t^*.t_i)$.
- $\forall a \in D(t^*).\mathsf{J}(t^*.a) = \mathsf{I}(t^*, a)$.

To briefly explain the $\mathcal{T}^*$, it provides a semantic mapping for every formula in unit graph expansion. Compared to the corresponding definition of domain functions and valuation frames back in Section 3, here the domain function cannot be determined by a sequence of literals. What was then a literal must be generalised to possibly conjunctively connected unit graph chains and literals. Note the implicit presumption of the associativity and commutativity of the classical $\wedge$ in the definition of $\dot{\mathcal{T}}$.

DEFINITION 14 (Valuation). *Suppose a valuation frame* $\mathfrak{M} = (\mathsf{I}, \mathsf{J})$*. The following are defined to hold.*

- $[\mathfrak{M} \models G] = \mathsf{J}(\textit{compress} \circ \textit{map}(G))$.

- $[\mathfrak{M} \models F_1 \wedge F_2] = [\mathfrak{M} \models F_1] \wedge^{\dagger} [\mathfrak{M} \models F_2]$.
- $[\mathfrak{M} \models F_1 \vee F_2] = [\mathfrak{M} \models F_1] \vee^{\dagger} [\mathfrak{M} \models F_2]$.

where map is defined by:

- $\textsf{map}(G_0 \wedge G_1) = \langle \textsf{map}(G_0), \textsf{map}(G_1) \rangle$.
- $\textsf{map}(G_0 \gtrdot G_1) = \lfloor \textsf{map}(G_0) \rfloor . \lfloor \textsf{map}(G_1) \rfloor$.
- $\textsf{map}(a) = \lfloor a \rfloor$.

and compress, given an input, returns the least element in the same congruence class as the input.

DEFINITION 15 (Validity and satisfiability). *A formula F with no occurrences of $\top$ and $\bot$ is said to be satisfiable in a valuation frame $\mathfrak{M}$ iff $[\mathfrak{M} \models F] = 1$; it is said to be valid iff it is satisfiable in all the valuation frames; it is said to be invalid iff it is not valid; and it is said to be unsatisfiable iff it is not satisfiable.*

I state the main results. Many details will be omitted, the proof approaches being similar to those that we saw in Section 3.

DEFINITION 16 (Procedure recursiveReduce2). *The procedure given below takes as an input a formula F in unit graph expansion.*

***Description of recursiveReduce2**(F)*

1. *Replace $\wedge$ and $\vee$ in F which is not in a chain with $\vee$ and respectively with $\wedge$. These two operations are simultaneous.*
2. *Replace all the non-chains $a \in \mathcal{A}$ in F simultaneously with a^c.*
3. *For every chain F_a in F which is not a strict a strict sub-chain of another chain, with its head F_h and its tail F_t, replace F_a with recursiveReduce2$(F_h) \vee (F_h \gtrdot$ recursiveReduce2$(F_t))$.*
4. *Reduce F via $\gtrdot$ reductions 4 and 5 in unit graph expansion.*

PROPOSITION 10 (Reduction of negated unit graph expansion). *Let F be a formula in unit graph expansion. Then $\neg F$ reduces via the $\neg$ and $\gtrdot$ reductions into* $\textsf{recursiveReduce2}(F)$ *which is in unit graph expansion. The reduction is unique.*

LEMMA 7 (Elementary complementation). *For any $G_0 \gtrdot G_1 \gtrdot \cdots \gtrdot G_k$ for $k \in \mathbb{N}$ and $G_k \in \mathcal{A}$,*[23] *if for a given valuation frame it holds that $[\mathfrak{M} \models G_0 \gtrdot G_1 \gtrdot \cdots \gtrdot G_k] = 1$, then it also holds that $[\mathfrak{M} \models \textsf{recursiveReduce2}(G_0 \gtrdot G_1 \gtrdot \cdots \gtrdot G_k)] = 0$; or if it holds that $[\mathfrak{M} \models \textsf{recursiveReduce2}(G_0 \gtrdot G_1 \gtrdot \cdots \gtrdot G_k)] = 1$, then it holds that $[\mathfrak{M} \models G_0 \gtrdot G_1 \gtrdot \cdots \gtrdot G_k] = 0$. These two events are mutually exclusive.*

PROOF. Let us abbreviate $\textsf{recursiveReduce2}$ by $\textsf{R}$. What we need to show for the first obligation is $[\mathfrak{M} \models \textsf{R}(G_0)] = [\mathfrak{M} \models G_0 \gtrdot \textsf{R}(G_1)] = \cdots = [\mathfrak{M} \models G_0 \gtrdot \cdots \gtrdot \textsf{R}(G_k)] = 0$. The reasoning process is recursive on each G_i, $0 \leq i \leq k$ within $\textsf{R}$. Since the formula size stays finite and since each reduction incurs finite branching, there is an end to each recursion. In the end, we will be showing that $[\mathfrak{M} \models G'_0 \gtrdot \ldots G'_{j-1} \gtrdot \textsf{R}(G'_j)] = 0$ for $j \in \mathbb{N}$ and $G'_j \in \mathcal{A}$, whenever the pattern is encountered during the recursion. For each such pattern, we will have that $[\mathfrak{M} \models G'_0 \gtrdot \ldots G'_{j-1} \gtrdot G'_j] = 1$ (the co-induction is left to readers; note the property of a formula in unit graph expansion). Then the result follows. $\dashv$

LEMMA 8 (Elementary double negation). *Let G denote $G_0 \gtrdot G_1 \gtrdot \cdots \gtrdot G_k$ for $k \in \mathbb{N}$ and $G_k \in \mathcal{A}$. Then for any valuation frame it holds that $[\mathfrak{M} \models G] = [\mathfrak{M} \models \textsf{recursiveReduce2}(\textsf{recursiveReduce2}(G))]$.*

PROOF. Let us use an abbreviation $\textsf{R}$ for $\textsf{recursiveReduce2}$ for space. $\textsf{R}(\textsf{R}(G)) = \textsf{R}(\textsf{R}(G_0) \vee (G_0 \gtrdot \textsf{R}(G_1)) \vee \cdots \vee (G_0 \gtrdot G_1 \gtrdot \cdots \gtrdot G_{k-1} \gtrdot \textsf{R}(G_k))) = \textsf{R}(\textsf{R}(G_0)) \wedge \textsf{R}(G_0 \gtrdot \textsf{R}(G_1)) \wedge \cdots \wedge \textsf{R}(G_0 \gtrdot G_1 \gtrdot \cdots \gtrdot G_{k-1} \gtrdot \textsf{R}(G_k))$. Since translation to disjunctive normal form is tedius, let us solve the problem directly

[23] G_k is always an element of $\mathcal{A}$ by the definition of a unit graph chain.

here. The strategy is that we first show $[\mathfrak{M} \models \mathsf{R}(\mathsf{R}(G_0))] = [\mathfrak{M} \models G_0]$, which reduces (via Lemma 7) the right hand side of the equation into $[\mathfrak{M} \models G_0 \wedge (G_0 \gtrdot \mathsf{R}(\mathsf{R}(G_1))) \wedge (G_0 \gtrdot \mathsf{R}(G_1 \gtrdot \mathsf{R}(G_2))) \wedge \cdots \wedge (G_0 \gtrdot \mathsf{R}(G_1 \gtrdot \cdots \gtrdot G_{k-1} \gtrdot \mathsf{R}(G_k)))] = [\mathfrak{M} \models (G_0 \gtrdot \mathsf{R}(\mathsf{R}(G_1))) \wedge (G_0 \gtrdot \mathsf{R}(G_1 \gtrdot \mathsf{R}(G_2))) \wedge \cdots \wedge (G_0 \gtrdot \mathsf{R}(G_1 \gtrdot \cdots \gtrdot G_{k-1} \gtrdot \mathsf{R}(G_k)))]$; we then show $[\mathfrak{M} \models G_0 \gtrdot \mathsf{R}(\mathsf{R}(G_1))] = [\mathfrak{M} \models G_0 \gtrdot G_1]$ to reduce again; and so on and so forth. In the end, we arrive at the required result. Therefore it suffices to show that $[\mathfrak{M} \models G_0 \gtrdot \cdots \gtrdot G_{i-1} \gtrdot \mathsf{R}(\mathsf{R}(G_i))] = [\mathfrak{M} \models G_0 \gtrdot \cdots \gtrdot G_{i-1} \gtrdot G_i], 0 \leq i \leq k$. But each G_i in $\mathsf{R}(\mathsf{R}(G_i))$ is strictly smaller in the number of symbols appearing within than G. So the reasoning is recursive. Because every formula is of a finite size and the reduction rules induce only finite branchings, it follows that every recursion is also finite, reaching at the obligation pattern of $[\mathfrak{M} \models G'_0 \gtrdot \cdots \gtrdot \mathsf{R}(\mathsf{R}(G'_j))] = [\mathfrak{M} \models G'_0 \gtrdot \cdots \gtrdot G'_j]$ for $G'_j \in \mathcal{A}$. But these equations hold by the way the local/global interpretations are defined. $\dashv$

THEOREM 3. *Denote by X the set of the expressions comprising all $[\mathfrak{M} \models G]$ for a formula G in unit graph expansion. Then for every valuation frame, $(X, \textsf{recursiveReduce2}, \dot{\top}, \dot{\bot}, \wedge^{\dagger}, \vee^{\dagger})$ with suppositional nullary connectives: $\dot{\top}$ and $\dot{\bot}$ defines a Boolean algebra.*

PROOF. It suffices to show anihilation, identity, associativity, commutativity, distributivity, idempotence, absorption, complementation and double negation. Straightforward with Lemma 7, Lemma 8, and by following the approaches taken in Section 3. $\dashv$

The insertion of the suppositional connectives into the theorem is inessential, since we could take it for granted that we are considering $\mathfrak{F}$ minus $\top$ and $\bot$ plus $\dot{\top}$ and $\dot{\bot}$, except that we never make use of $\dot{\top}$ or $\dot{\bot}$ in an expression. For

the following results, let us enforce that $\mathcal{G}(F)$ denote the set of formulas in unit graph expansion that F without the occurrences of $\top$ and $\bot$ reduce into.

THEOREM 4 (Bisimulation). *Assumed below are pairs of formulas in which $\top$ and $\bot$ do not occur. F' differs from F only by the shown sub-formulas,* i.e. *F' derives from F by replacing the shown sub-formula for F' with the shown sub-formula for F and vice versa. Then for each pair (F, F') below, it holds for every valuation frame that $[\mathfrak{M} \models F_1] = [\mathfrak{M} \models F_2]$ for all $F_1 \in \mathcal{G}(F)$ and for all $F_2 \in \mathcal{G}(F')$.*

$$F[G_0 \vee \cdots \vee G_{k+1} \gtrdot F_c]\,,\, F'[\bigvee_{I \in \mathcal{I}(k+1)} (\bigwedge_{j \in I} G_j \gtrdot F_c)]$$

$$F[F_a \gtrdot F_b \wedge F_c]\,,\, F'[(F_a \gtrdot F_b) \wedge (F_a \gtrdot F_c)]$$

$$F[F_a \gtrdot F_b \vee F_c]\,,\, F'[(F_a \gtrdot F_b) \vee (F_a \gtrdot F_c)]$$

$$F[F_a \wedge (F_b \vee F_c) \gtrdot F_d]\,,\, F'[(F_a \wedge F_b) \vee (F_a \wedge F_c) \gtrdot F_d]$$

$$F[(F_a \vee F_b) \wedge F_c \gtrdot F_d]\,,\, F'[(F_a \wedge F_c) \vee (F_b \wedge F_c) \gtrdot F_d]$$

$$F[\neg G]\,,\, F'[\textit{recursiveReduce2}(G)]$$

$$F[\neg(F_a \wedge F_b)]\,,\, F'[\neg F_a \vee \neg F_b]$$

$$F[\neg(F_a \vee F_b)]\,,\, F'[\neg F_a \wedge \neg F_b]$$

$$F[\neg(G_0 \wedge \ldots G_k \gtrdot F_a)]\,,\, F'[\neg G_0 \vee \cdots \vee \neg G_k \vee (G_0 \wedge \cdots \wedge G_k \gtrdot \neg F_a)]$$

$$F[F_a \vee F_a]\,,\, F'[F_a]$$

$$F[F_a \wedge F_a]\,,\, F'[F_a]$$

$$F[F_a \vee F_b]\,,\, F'[F_b \vee F_a]$$

$$F[F_a \vee (F_b \vee F_c)]\,,\, F'[(F_a \vee F_b) \vee F_c]$$

$$F[F_a \wedge F_b]\,,\, F'[F_b \wedge F_a]$$

$$F[F_a \wedge (F_b \wedge F_c)]\,,\, F'[(F_a \wedge F_b) \wedge F_c]$$

PROOF. Similar in approach to the proof of Theorem 2, the proof following by simultaneous composite induction by the induction measure as in Proposition 9, which strictly decreases at each reduction. We first establish that $\mathcal{G}(F_1) = \mathcal{G}(F_2)$. One way, to show that to each reduction on F' corresponds reduction(s) on F, is straightforward, for we can choose to reduce F into F' in most of the cases, thereafter synchronizing both of the reductions. We show one sub-proof for the 10th pair, however, to be very safe. If $\gg$ reduction 3 applies ($F_a = G$ for some unit graph chain G), then we have; $F[G_0 \vee G_1 \vee \cdots \vee (G \vee G) \vee \cdots \vee G_{k+1} \gg F_a] \mapsto F_p[\bigwedge_{I \in \mathcal{I}(k+1)}(\bigvee_{j \in I} G_j \gg F_a)]$. Since association by $\vee$ on the constituting unit graph chains is freely chosen, let us assume that $\bigwedge_{I \in \mathcal{I}(k+1)}(\bigvee_{j \in I} G_j \gg F_a)$ is ordered such that, from the left to the right, the number of the occurrences of G either stays the same, or else strictly increases. Then, generally speaking, there are three groups of sub-formulas: those that do not have the occurrences of the G; those in which the G occurs once; and those in which there are two occurrecnes of G. Now, for F', applying the same reduction rule, we gain: $F'[G_0 \vee G_1 \vee \cdots \vee G \vee \cdots \vee G_k \gg F_a] \mapsto F'_p[\bigwedge_{I \in \mathcal{I}(k)}(\bigvee_{j \in I} G_j \gg F_a)]$. It is straightforward to see that F'_p involves every constituent from the first group of F_p (if the group has any constituent at all); and half of the constituents from the second group. So we can sequentially apply induction hypothesis on F_p to match up with F'_p. Induction hypothesis.

Into the other way to show that to each reduction on F corresponds reduction(s) on F':

1. The first pair:
   (a) If a reduction takes place on a sub-formula which neither is a sub-formula of the shown sub-formula nor takes as its sub-formula the

shown sub-formula, then we reduce the same sub-formula in F'. Induction hypothesis (note that the number of reduction steps is that of F into this direction).

(b) If it takes place on a sub-formula of F_c then we reduce the same sub-formula of both occurrences of F_c in F'. Induction hypothesis.

(c) If $\gtrdot$ reduction 3 takes place on F such that we have; $F[G_0 \vee \cdots \vee G_{k+1} \gtrdot F_c] \rightsquigarrow F_x[\bigvee_{I \in \mathcal{I}(k+1)}(\bigwedge_{j \in I} G_j \gtrdot F_c)]$, F and F_x differ only by the shown sub-formulas. Do nothing on F', and $F_x = F'$. Vacuous thereafter.

(d) If a reduction takes place on a sub-formula F_p of F in which the shown sub-formula of F occurs as a strict sub-formula ($F[G_0 \vee \cdots \vee G_{k+1} \gtrdot F_c] = F[F_p[G_0 \vee \cdots \vee G_{k+1} \gtrdot F_c]]$), then we have $F[F_p[G_0 \vee \cdots \vee G_{k+1} \gtrdot F_c]] \rightsquigarrow F_x[F_q[G_0 \vee \cdots \vee G_{k+1} \gtrdot F_c]]$. But we have $F' = F'[F'_p[\bigvee_{I \in \mathcal{I}(k)}(\bigwedge_{j \in I} G_j \gtrdot F_c)]]$. Therefore we apply the same reduction on F'_p to gain;
$F'[F'_p[\bigvee_{I \in \mathcal{I}(k)}(\bigwedge_{j \in I} G_j \gtrdot F_c)]] \rightsquigarrow F'_x[F'_{p'}[\bigvee_{I \in \mathcal{I}(k)}(\bigwedge_{j \in I} G_j \gtrdot F_c)]]$. Induction hypothesis.

2. The second and the third: Straightforward.

3. The fourth pair:

(a) If obj distribution 1 takes place on F such that we have; $F[F_a \wedge (F_b \vee F_c) \gtrdot F_d] \rightsquigarrow F_x[(F_a \wedge F_b) \vee (F_a \wedge F_c) \gtrdot F_d]$, then do nothing on F'; and we have $F_x = F'$. Vacuous thereafter.

(b) If obj distribution 2 takes place on F such that we have; $F[(F_\beta \vee F_\gamma) \wedge (F_b \vee F_c) \gtrdot F_d] \rightsquigarrow F_x[(F_\beta \wedge (F_b \vee F_c)) \vee (F_\gamma \wedge (F_b \vee F_c)) \gtrdot F_d]$ for $F_a = F_\beta \vee F_\gamma$, then by induction hypothesis, it does not cost generality if we replace it with $F_p[((F_\beta \wedge F_b) \vee (F_\beta \wedge F_c)) \vee ((F_\gamma \wedge F_b) \vee (F_\gamma \wedge F_c)) \gtrdot F_d]$ that differs from F_x only by the shown sub-formulas. Meanwhile, we derive; $F'[((F_\beta \vee F_\gamma) \wedge F_b) \vee ((F_\beta \vee F_\gamma) \wedge F_c) \gtrdot F_d] \rightsquigarrow F'_x[((F_\beta \wedge F_b) \vee$

$(F_\gamma \wedge F_b)) \vee ((F_\beta \wedge F_c) \vee (F_\gamma \wedge F_c)) \gtrdot F_d]$. Apply induction hypothesis on F_x to arrive at F'_x. Vacuous thereafter.

   (c) The other cases: Straightforward.

4. The fifth pair: Similar.
5. The sixth pair: Straightforward, since a unit graph chain cannot be further reduced, since the reduction of $\neg G$ is unique, and since, by the definition of $\gtrdot$ reduction 3, it cannot apply unless $\neg G$ has been fully reduced to $\mathsf{recursiveReduce2}(G)$.
6. The rest: Cf. the proof of Theorem 2.

By the result of the above bisimulation, we now have $\mathcal{G}(F) = \mathcal{G}(F')$. However, it takes only those 4 $\neg$ reductions, 3 $\gtrdot$ reductions, obj distribution 1 and obj distribution 2 to derive a formula in unit graph expansion; hence we in fact have $\mathcal{G}(F) = \mathcal{G}(F_x)$ for some formula F_x in unit graph expansion. But then by Theorem 3, there could be only one value out of $\{0, 1\}$ assigned to $[\mathfrak{M} \models F_x]$, as required. $\dashv$

COROLLARY 2 (Normalisation). *Given a formula F with no occurrences of $\top$ and $\bot$, denote the set of formulas in unit graph expansion that it can reduce into by $\mathcal{G}_1$. Then it holds for every valuation frame either that $[\mathfrak{M} \models F_a] = 1$ for all $F_a \in \mathcal{G}_1$ or else that $[\mathfrak{M} \models F_a] = 0$ for all $F_a \in \mathcal{G}_1$.*

By construction, this logic is also decidable, but for any unrestrained expressions the complexity will be high, owing to $\gtrdot$ reduction 3.

§6. Conclusion. This work analysed phenomena that arise around concepts and their attrbutes, and called attention the positioning of atomic entities in formal logic, based on the notion of recognition cut-off. Both the philosophical and the mathematical foundations of gradual logic were laid down. For the object-attribute relation, there may be many linguistically reasonable interpretations.

To conclude, I state connections of gradual logic to Aristotle's logic and others, along with prospects.

6.1. Connection to Aristotle's logic. Natural expressions require more than one types of negation. Given an expression X, if it is contradictorily negated into another expression Y, X is true iff Y is false. A contrary expression to X, however, only demands that it be false if X is true. A sub-contrary expression to X, on the other hand, demands that it be true if X is false. The distinction, which is as good as defunct in the post-Fregean modern logic, despite sporadic recurrences of the theme here and then [25, 28, 20], has been nonetheless known - already since the era of Aristotle's. An extensive discussion on contrarieties is found in Categories for qualities and in Prior Analytics for categorical sentences. Meanwhile, only external, contradictory negations remain proper in the modern logic, contrarieties dismissed. Some such as Lukasiewicz [19] contend that the modern logic on sentences, founded by the Stoics and axiomatized by Frege, is logically prior to Aristotelian term logic in that, according to his judgement, logic on propositions underlies the term logic. Others are not so convinced. Some such as Horn, Sommers and Englebretsen defend the term logic in the respective works of theirs [16, 26, 7], countenancing that there are features that have been lost in the mass-scale migration from the term logic to the modern logic. With analysis in Section 1, I have in part concurred with proponents of the term logic. One that is of particular interest in the Aristotelian term logic is the use of indefinite[24] nouns as result from prefixing 'not' to a noun, *e.g.* from man to not-man; and of indefinite verbs, from walks to not-walks. For sentences, they can be either affirmative or negative. "Man walks", "Not-Man walks", "Man Not-walks", and "Not-Man Not-walks" are affirmative; "Man does not

[24] These terminologies are taken from The Internet Classics Archive (classics.mit.edu/Aristotle).

Not-walk", "Not-Man does not walk", "Man does not Not-walk", and "Not-Man does not Not-walk" are negative corresponding to them.[25] Hence to each singular sentence[26], say "Man walks", there are seven corresponding sentences that are either negative or having an indefinite term. There is no such denying particle on terms in Fregean modern logic; and diversification of negation by scopal distinction [25] or by adopting more than one external negation operators [28] on sentences cannot make amends for the limitation that arises directly from sentential atomicity. One crucial point about a proposition in the modern logic in fact is that, even if it is an atomic proposition, one never knows how complex it already is, and consequently how many contraries it ought to have (Cf. also Geach [10]). However, concerning this matter, the assumption of atomicity of entities in formal/symbolic logic may be more fundamental. Although to Aristotle, too, there existed indivisible entities, it is unlikely that such entities, if they are to exist, are cognizable, so long as the entities that we deal with are concepts that refer to objects; and, about those, we cannot reason. One may also apply Postulate 1 to propositions in general, whereby a proposition becomes an object, which will then be divisible. Then there will be propositions about the proposition as its attributes. By explaining the indivisible in terms of recognition cut-offs, one can also appreciate that any proposition, if treated as an object, will have attributive propositions about it, and they, too, as internal

[25] Judging from Aristotle's texts (in translation), affirmative sentences in the form: X is Y, are primary to Aristotle. X and Y can be in the form Not-X' or Not-Y', for they are in any case affirmed in the sentence. However, it is not the case for Aristotle that sentences of the form: X is not Y, denying Y of X, bears a truth value primarily. This is clear from an example (Cf. On Interpretation) about the truth value of 'Socrates is ill/Socrates is not ill'. In case Socrates does not denote anything, then the non-being (and non-being is not a being) cannot be ill in an ordinary sense, and so 'Socrates is ill' is false. Aristotle then judges that 'Socrates is not ill' is true. But this would remain debatable if it were the case that negative statements primarily bore a truth value. For, then, it, by exactly the same reasoning, could be simply false in the absence of Socrates. Therefore, while in the Fregean logic whatever sentences may be a proposition, with no regard whether it is affirmative or negative, the differentiation is important in Aristotelian term logic.

[26] A sentence that does not specify "all" or "some" are singular.

structures of the proposition, can be structured in $\gtrdot$s. Then not just one, not just two, but arbitrarily many internal contradictions can be brought to light, which are externally contraries, for those propositions that have been hitherto atomic.

There is another relevant remark of Aristotle's found in Prior/Posterior Analytics. It is combinability of predicates. If bird is for example both beautiful and singing, then that could be expressed in the first-order logic as; $\mathsf{IsBeautiful}(bird) \wedge \mathsf{IsSinging}(bird)$, where $bird$ is a term. But in so expressing, it goes no further. When we attempt a proximity mapping of 'Bird is beautiful, and it is singing' in gradual logic with 'Bird is (judged under a domain of discourse); and it has the attribute of being beautiful, and that it has the capacity of singing (judged under another domain of discourse),' we gain; $(\mathsf{Bird} \gtrdot \mathsf{Beautiful}) \wedge (\mathsf{Bird} \gtrdot \mathsf{Singing})$; or, equivalently $\mathsf{Bird} \gtrdot \mathsf{Beautiful} \wedge \mathsf{Singing}$, the two attributes conjoining into a unified attribute. Similar may also hold for the other side of $\gtrdot$. Exactly how combination occurs depends on a given linguistic interpretation on the object-attribute relation. Aristotle mentions of such combination in one part. Now, why a similar process does not occur in the above-given first-order expression is because, not only of the terms but also of all the predicates, form is pre-defined. Fregean terms are indivisible and fixed; and it must be known how many Fregean terms each Fregean predicate will take.

To conclude this sub-section, we saw that first-order logic does not share the same logical foundation of the term logic. Aristotle's logic treats terms both as subjects and predicates, whereas Fregean terms are not Fregean predicates, nor vice versa. The object-attribute relation in gradual logic is closer in the respect to Aristotle's subject-predicate relation than the relation that holds between Fregean terms and Fregean predicates. As stated in 5.1, however, in gradual logic $X \gtrdot Y$ may itself act as an object, as in $(X \gtrdot Y) \gtrdot Z$, distinct from Aristotelian subject-predicate relation which does not produce a subject.

As one research interest out of gradual logic, it should be fruitful to conduct cross-studies against Aristotle's logic, and to see how well Aristotle's syllogism can be explained within. Instead of embedding Aristotle's logic in first-order logic [1] or first-order logic in Aristotle's logic [26], the strengths that the two have may be mutually extended. gradual logic may pave a way for realising the possibility. It must be pointed out, however, that, in order to attempt modelling the universal/particular sentences, the three figures and syllogism in Prior Analytics, it is necessary that we develop gradual predicate logic, as to be stated shortly.

6.2. Gradual X Logic. Meta-framework of some existing framework(s) offers a way of deriving new results without destroying the properties of the original framework(s). As it retains principles in the original framework, it is also highly reusable. For example, assuming that all the (Fregean) terms and quantifications are contained within a domain of discourse, replacement of the underlying propositional logic in this work with first-order logic, or, in general, another Boolean logic X, gives us gradual X logic, and all the main results that we saw go through, apart from the decidability result which depends on the decidability of the underlying logic. The reason that we can simply swap the underlying in this manner is because the meta-framework considered in this work acts only on the 0/1 (Cf. the given semantics). How the 0/1 is generated is irrelevant to the applicability of the meta-constructs that $\gtrdot$s generate.

However, from a theoretical perspective about the use of predicates within gradual logic, the use of intra-domain-of-discourse predicates/quantifications is conservative, since the terms so introduced will be atomic, which is not in harmony with the philosophical standpoint that was taken in this work. Real theoretical extensions will be by introducing predicates that range over attributed objects themselves. For instance, suppose that we have two expressions $\mathsf{Adjective} \gtrdot$

Sheep and Ovine, then we may say $\mathsf{IsEqualTo}(\mathsf{Adjective} \gg \mathsf{Sheep}, \mathsf{Ovine})$ in some (but not necessarily all) domains of discourse. This type of extension may be called active predicate extension. In this direction, there are both philosophical/linguistic and mathematical challenges, and it will be important to adequately capture interactions between the active predicates and the reduction rules.

6.3. On tacit agreement. The incremental shift in domain of discourse models tacit agreement, which is otherwise understood as a context. Within formal logic considered in artificial intelligence, a line of studies since McCarthy [21, 4, 11, 24] have set a touchstone for logics handling contextual reasonings. In those context logics,[27] all the propositions are judged under a context depending on which their truth values are determined. The question of what a context is, nonetheless, has not been pursued in the context logics any farther than that it is a rich object that is only partially explained. But because they treat a proposition as, in comparison, something that is known, there emerges a distinction between a context and a proposition whereby the former becomes a meta-term like a nominal in hybrid logic [3] that conditions the latter. As much as the consideration appears natural, it may be also useful to think what truly makes a context differ from a proposition, for, suppose a proposition that Holmes LS is a detective in the context of Sherlock Holmes stories [21], it appears on a reasonable ground that that the scenes (under which the proposition falls) are the stories of Sherlock Holmes is indeed a proposition. And if a proposition itself is a rich object that can be only partially explained, then the fundamental gap between the two domains closes in.

[27] There are other logics termed context logics which treat a context as an implication. But these, by explicitly stating what follow from what in the same domain of discourse, do not truly express the tacitness of a tacit agreement.

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Appendix A - a Java file, and test cases for Proposition 7. The Java code (version 1.6) that is used for the tests of Proposition 7 and Proposition 9 is listed below.

```
import java.util.Stack;
/**
 * Written just for proving two results on "Logic on
 * Recognition Cut-Off: Objects, Attributes and Atomicity".
 * This program does not perform any value comparisons, which must be manually done.
 * Intended for a personal use, there are hardly any exception handling. In any case,
 * the source is in public to see where errors are thrown.
 * An argument should be provided. If it is 1, then the program checks the results
```

```
 * for Proposition 7(Preliminary observation); otherwise, it checks the results
 * for Proposition 9.
 * The outputs look like this: <p>
 * "======TEST n, X ======== first expression === second expression"
 * on the first line where the 'n' indicates which test procedure that is being
 * called; X is either true or false corresponding to the Boolean parameter of
 * f_size function on the paper; the 'first expression' and the 'second
 * expression' a formula whose value is being calculated. <p>
 * On the second line appears the calculated result of the 'first expression'.<p>
 * And on the third line that of the 'second expression'.<p>
 * On the fourth line are found for the 'first expression' the inverse of
 * ((the number of occurrences of !) + 1), that of % (+ 1 on the denumerator),
 * and, in case of the second test, also that of * (+ 1 on the denumerator). <p>
 * ON the fifth line, same but for the 'second expression'. <p>
 * This basic structure repeats as many as the number of the test cases. <p>
 * Briefly remarking on the syntax, an expression is written in prefix form.
 * A small alphabet which must be of length 1 denotes a literal. A capital
 * alphabet of the length 1 denotes a general formula, on the other hand.
 * Grammar (let us denote an expression by EXP): <p>
 * 1. A formula is EXP.<p>
 * 2. !EXP is EXP. ! means not. <p>
 * 3. *(EXP)(EXP) is EXP. * means and. <p>
 * 4. +(EXP)(EXP) is EXP. + means or. <p>
 * 5. %(EXP)(EXP) is EXP. % means the object-attribute relation. <p>
 * Also remarking on the results, ^n is exponent to immediately preceding number.
 * n/m denotes n divided by m. + denotes addition.
 * (l=n) after a capital alphabet indicates the value to the second argument of
 * the function f_size.
 * @author
 *
 */
public class Calculator {

	//========DATA========
	private static enum FType{
		LITERAL,	//a literal.
		NONLITERAL,	//not a literal.
		GFORMULA	//Sub-formula still to be processed.

	}
```

```java
private static String f_left;
private static String f_right;

private static Stack<Integer> stack;

//these three are used to get the number of occurrences of !,% and *.
private static int neg_counter;
private static int obat_counter;
private static int and_counter;
//===================
/**
 * If args = "1", this procedure checks the results for Proposition 7. At the
 * same time, it tells the inverse of the number of occurrences of negations
 * (!) and the inverse of the number of occurrences of .> (%).
 * Otherwise, it checks the results for Proposition 9. At the same time, it
 * tells the inverse of the number of the occurrences of negations (!), the
 * inverse of the number of the occurrences of .> (%),
 * and the inverse of the number of the occurrences of conjunctions (*).
 * @param args
 */
public static void main(String[] args){

   if(args[0].equals("1"))
   {
        test1(); test2();  test3(); test4();test5();test6();test7();
        test8();test9();test10();test11();test12();test13();
   }
   else
   {
          test1B();
          test2B();test3B();test4B();test5B();test6B();test7B();test8B();
          test9B();test10B();test11B();test12B();test13B();
   }
}
/**
 * Test neg reduction 1.
 */
private static void test1(){
        f_left = new String("!s");
        f_right = new String("s");
```

```
        printOut(f_left,f_right,1);

}
/**
 * Test neg reduction 2.
 */
private static void test2(){
        f_left = new String("!*(A)(B)");
        f_right = new String("+(!A)(!B)");
        printOut(f_left, f_right,2);
}
/**
 * Test neg reduction 3.
 */
private static void test3(){
        f_left = new String("!+(A)(B)");
        f_right = new String("*(!A)(!B)");
        printOut(f_left, f_right,3);
}
/**
 * Test neg reduction 4.
 */
private static void test4(){
        f_left = new String("!%(s)(A)");
        f_right = new String("+(s)(%(s)(!A))");
        printOut(f_left, f_right, 4);
}
/**
 * Test neg reduction 5.
 */
private static void test5(){
        f_left = new String("%(%(A)(B))(C)");
        f_right = new String("*(%(A)(C))(+(%(A)(B))(%(A)(%(B)(C))))");
        //(f_left, f_right, 5);
        printOut(f_left,f_right,5);

}
/**
 * Test .> reduction 1.
 */
```

```
private static void test6(){
        f_left = new String("%(*(A)(B))(C)");
        f_right = new String("*(%(A)(C))(%(B)(C))");
        printOut(f_left,f_right,6);
}
/**
 * Test .> reduction 2.
 */
private static void test7(){
        f_left = new String("%(+(A)(B))(C)");
        f_right = new String("+(%(A)(C))(%(B)(C))");
        printOut(f_left,f_right,7);

}
/**
 * Test .> reduction 3.
 */
private static void test8(){

        f_left = new String("%(A)(*(B)(C))");
        f_right = new String("*(%(A)(B))(%(A)(C))");
        printOut(f_left,f_right,8);
}
/**
 * Test .> reduction 4.
 */
private static void test9(){
        f_left = new String("%(A)(+(B)(C))");
        f_right = new String("+(%(A)(B))(%(A)(C))");
        printOut(f_left,f_right,9);
}
/**
 * Test * commutativity.
 */
private static void test10(){
        f_left = new String("*(A)(B)");
        f_right = new String("*(B)(A)");
        printOut(f_left,f_right,10);
}
/**
```

```
 * Test + commutativity.
 */
private static void test11(){
        f_left = new String("+(A)(B)");
        f_right = new String("+(B)(A)");
        printOut(f_left,f_right,11);
}
/**
 * Test * associativity.
 */
private static void test12(){
        f_left = new String("*(*(A)(B))(C)");
        f_right = new String("*(A)(*(B)(C))");
        printOut(f_left,f_right,12);
}
/**
 * Test + associativity.
 */
private static void test13(){
        f_left = new String("+(+(A)(B))(C)");
        f_right = new String("+(A)(+(B)(C))");
        printOut(f_left,f_right,13);
}
/**
 * Tests for Proposition 9. ! reduction 1.
 */
private static void test1B(){
        test1();
}
/**
 * For ! reduction 2.
 */
private static void test2B(){
        test2();
}
/**
 * For ! reduction 3.
 */
private static void test3B(){
        test3();
```

```
}
/**
 * For ! reduction 4, for k =2. Association does not matter
 * due to other cases in the same proposition.
 */
private static void test4B(){
        f_left = new String("!%(*(A)(*(B)(C)))(D)");
        f_right = new String("+(+(!A)(+(!B)(!C)))(%(*(A)(*(B)(C)))(!D))");
        printOut(f_left, f_right, 400);
}
/**
 * For .> reduction 3, for k = 2.
 */
private static void test5B(){
        f_left = new String("%(+(A)(+(B)(C)))(D)");
        f_right = new String("+(%(A)(D))(+(%(B)(D))(+(%(C)(D))(+(%(*(A)(B))(D))"+
        "(+(%(*(A)(C))(D))(+(%(*(B)(C))(D))(%(*(A)(*(B)(C)))(D)))))))");
        printOut(f_left,f_right,500);
}
/**
 * For .> reduction 4.
 */
private static void test6B(){
        test8();
}
/**
 * For .> reduction 5.
 */
private static void test7B(){
        test9();
}
/**
 * For obj distribution 1.
 */
private static void test8B(){
        f_left = new String("%(*(A)(+(B)(C)))(D)");
        f_right = new String("%(+(*(A)(B))(*(A)(C)))(D)");
        printOut(f_left,f_right,800);
}
/**
```

```
 * For obj distribution 2.
 */
private static void test9B(){
        f_left = new String("%(*(+(A)(B))(C))(D)");
        f_right = new String("%(+(*(A)(C))(*(B)(C)))(D)");
        printOut(f_left,f_right,900);
}
/**
 * For * commutativity.
 */
private static void test10B(){
        test10();
}
/**
 * For + commutativity.
 */
private static void test11B(){
        test11();
}
/**
 * For * associativity.
 */
private static void test12B(){
        test12();
}
/**
 * For + associativity.
 */
private static void test13B(){
        test13();
}
/**
 * As on the paper, save in the prefix form. XX indicates that the string is
 *  not formulated according to the grammar.
 * @param neg_depth
 * @param l
 * @param bool
 * @param f_str
 * @return
 */
```

```
private static String f_size(int neg_depth, int l, boolean bool, String f_str){
        String prefix;
        String suffix;
        if(getFType(f_str) == FType.LITERAL)
                return "1/4^" + (new Integer(l).toString());
        else if(getFType(f_str) == FType.NONLITERAL)
                return f_str+ "(l=" + new Integer(l).toString()+ ")";
        else if(f_str.charAt(0) == '!')
        {
                negCounterIncrement();
                return "1/4^" + (new Integer(neg_depth).toString()) +  " + (" +
                                f_size(neg_depth,l, bool, f_str.substring(1))+ ")";
        }
        else if(f_str.charAt(0) == '%')
        {
                obatCounterIncrement();
                int index = getIndexParenthesis(f_str);
                prefix = f_str.substring(2, index);
                suffix = f_str.substring(index+2,f_str.length()-1);
                return "(" +
                        f_size(neg_depth+1,l, bool, prefix) + " + "
                                + f_size(neg_depth+1,l,bool, suffix) + ")";

        }

        else if(f_str.charAt(0) == '*')
        {
                andCounterIncrement();
                int index = getIndexParenthesis(f_str);
                prefix = new String(f_str.substring(2, index));
                suffix = new String(f_str.substring(index+2,f_str.length()-1));
                if(bool) return "max(" + f_size(neg_depth +1,l+1,false,prefix) +
                                " , " + f_size(neg_depth+1,l+1,false,suffix) + ")";
                else return "max(" + f_size(neg_depth,l,false,prefix) +
                                " , " + f_size(neg_depth,l,false,suffix) + ")";

        }
        else if(f_str.charAt(0)=='+'){
                int index = getIndexParenthesis(f_str);
                prefix = f_str.substring(2, index);
```

```
            suffix = f_str.substring(index+2,f_str.length()-1);
            if(bool) return "max(" + f_size(neg_depth+1,l+1,false,prefix) +
                ", " + f_size(neg_depth+1,l+1,false,suffix) + ")";
            else return "max(" + f_size(neg_depth,l,false,prefix) +
                        ", " + f_size(neg_depth,l,false,suffix) + ")";
        }

        return "XX";
}
/**
 * Given a string, it tells if it is a literal or a general formula, or
 * otherwise.
 * @param in_str
 * @return
 */
private static FType getFType(String in_str)
{
        if(in_str.length() >= 2)
        {
                //System.out.println("Getting GFORMULA which is: " + in_str);
                return FType.GFORMULA;
        }
        else if(in_str.charAt(0) >= 'a' &&
                        in_str.charAt(0) <= 'z')
                return FType.LITERAL;
        else
        {
        //      System.out.println(in_str + " Is NONLITERAL.");
                return FType.NONLITERAL;
        }

}
/**
 * Parser. -100 is an error.
 * @param in_str
 * @return
 */
private static Integer getIndexParenthesis(String in_str)
{
        //with stack.
```

```
        stack = new Stack<Integer>();
        String curStr = new String(in_str);
        char curChar;
        for(int i =0; i <curStr.length(); i++)
        {
                curChar = curStr.charAt(i);
        if (curChar == '(')
                        stack.push(i);
                else if (curChar == ')')
                {
                        stack.pop();
                        if(stack.isEmpty())
                                return i;
                }

        }

        return -100;

}
/**
 * Console printing.
 * @param left_str
 * @param right_str
 * @param n
 */
private static void printOut(String left_str, String right_str, int n)
{
        int lNegCounter,lObatCounter,lAndCounter;
        counterReset();
        System.out.println("=====TEST" + new Integer(n).toString() +
            ", false=======" + left_str + "===" + right_str);
        System.out.println(f_size(1,1, false, f_left));
        //store the counter values for the first expression.
        lNegCounter = neg_counter; lObatCounter = obat_counter; lAndCounter =
                        and_counter;
        //and reset the counters.
        counterReset();
        System.out.println(f_size(1,1,false,f_right));
        //print the counter values.
```

```
        System.out.println("1/" + lNegCounter + ",1/" + lObatCounter + ",1/"
        + lAndCounter);
        System.out.println("1/" + neg_counter + ",1/" + obat_counter + ",1/"
        + and_counter);

        counterReset();
        System.out.println("=====TEST" + new Integer(n).toString() + ", " +
                        "true=======");
        System.out.println(f_size(1,1,true,f_left));
        lNegCounter = neg_counter; lObatCounter = obat_counter;
        lAndCounter = and_counter;
        counterReset();
        System.out.println(f_size(1,1,true,f_right));
        System.out.println("1/" + lNegCounter + ",1/" + lObatCounter + ",1/" +
        lAndCounter);
        System.out.println("1/" + neg_counter + ",1/" + obat_counter + ",1/" +
        and_counter);
}
/**
 * Reset the counters to 1. 1 means basically 0, but as mentioned in the
 * class description, 1/0 is bad. So the minimum is 1.
 */
private static void counterReset()
{
        obat_counter = 1; neg_counter = 1; and_counter = 1;
}
/**
 * Increases the counter counting the occurrences of %.
 */
private static void obatCounterIncrement()
{
        obat_counter++;
}
/**
 * Increases the counter counting the occurrences of !.
 */
private static void negCounterIncrement()
{
        neg_counter++;
}
```

```
    /**
     * Increases the counter counting the occurrences of *.
     */
    private static void andCounterIncrement()
    {
        and_counter++;
    }
}
```

And the test cases for Proposition 7 below. Please refer to the class description of the Java source code for the format. Test 1 tests $\neg$ reduction 1, Test 2 tests $\neg$ reduction 2, and so on until Test 4. Test 5 tests $\gtrdot$ reduction 1, Test 6 does $\gtrdot$ reduction 2, and so on until Test 9. Test 10 and 11 test commutativity of $\wedge$ and $\vee$. Test 12 and Test 13 associativity.

```
=====TEST1, false=======!s===s
1/4^1 + (1/4^1)
1/4^1
1/2,1/1,1/1
1/1,1/1,1/1
=====TEST1, true=======
1/4^1 + (1/4^1)
1/4^1
1/2,1/1,1/1
1/1,1/1,1/1
=====TEST2, false=======!*(A)(B)===+(!A)(!B)
1/4^1 + (max(A(l=1) , B(l=1)))
max(1/4^1 + (A(l=1)) , 1/4^1 + (B(l=1)))
1/2,1/1,1/2
1/3,1/1,1/1
=====TEST2, true=======
1/4^1 + (max(A(l=2) , B(l=2)))
max(1/4^2 + (A(l=2)) , 1/4^2 + (B(l=2)))
1/2,1/1,1/2
1/3,1/1,1/1
=====TEST3, false=======!+(A)(B)===*(!A)(!B)
1/4^1 + (max(A(l=1) , B(l=1)))
max(1/4^1 + (A(l=1)) , 1/4^1 + (B(l=1)))
1/2,1/1,1/1
1/3,1/1,1/2
=====TEST3, true=======
1/4^1 + (max(A(l=2) , B(l=2)))
```

```
max(1/4^2 + (A(l=2)) , 1/4^2 + (B(l=2)))
1/2,1/1,1/1
1/3,1/1,1/2
=====TEST4, false=======!%(s)(A)===+(s)(%(s)(!A))
1/4^1 + ((1/4^1 + A(l=1)))
max(1/4^1 , (1/4^1 + 1/4^2 + (A(l=1))))
1/2,1/2,1/1
1/2,1/2,1/1
=====TEST4, true=======
1/4^1 + ((1/4^1 + A(l=1)))
max(1/4^2 , (1/4^2 + 1/4^3 + (A(l=2))))
1/2,1/2,1/1
1/2,1/2,1/1
=====TEST5, false=======%(%(A)(B))(C)===*(%(A)(C))(+(%(A)(B))(%(A)(%(B)(C))))
((A(l=1) + B(l=1)) + C(l=1))
max((A(l=1) + C(l=1)) , max((A(l=1) + B(l=1)) , (A(l=1) + (B(l=1) + C(l=1)))))
1/1,1/3,1/1
1/1,1/5,1/2
=====TEST5, true=======
((A(l=1) + B(l=1)) + C(l=1))
max((A(l=2) + C(l=2)) , max((A(l=2) + B(l=2)) , (A(l=2) + (B(l=2) + C(l=2)))))
1/1,1/3,1/1
1/1,1/5,1/2
=====TEST6, false=======%(*(A)(B))(C)===*(%(A)(C))(%(B)(C))
(max(A(l=1) , B(l=1)) + C(l=1))
max((A(l=1) + C(l=1)) , (B(l=1) + C(l=1)))
1/1,1/2,1/2
1/1,1/3,1/2
=====TEST6, true=======
(max(A(l=2) , B(l=2)) + C(l=1))
max((A(l=2) + C(l=2)) , (B(l=2) + C(l=2)))
1/1,1/2,1/2
1/1,1/3,1/2
=====TEST7, false=======%(+(A)(B))(C)===+(%(A)(C))(%(B)(C))
(max(A(l=1) , B(l=1)) + C(l=1))
max((A(l=1) + C(l=1)) , (B(l=1) + C(l=1)))
1/1,1/2,1/1
1/1,1/3,1/1
=====TEST7, true=======
(max(A(l=2) , B(l=2)) + C(l=1))
```

```
max((A(l=2) + C(l=2)) , (B(l=2) + C(l=2)))
1/1,1/2,1/1
1/1,1/3,1/1
=====TEST8, false=======%(A)(*(B)(C))===*(%(A)(B))(%(A)(C))
(A(l=1) + max(B(l=1) , C(l=1)))
max((A(l=1) + B(l=1)) , (A(l=1) + C(l=1)))
1/1,1/2,1/2
1/1,1/3,1/2
=====TEST8, true=======
(A(l=1) + max(B(l=2) , C(l=2)))
max((A(l=2) + B(l=2)) , (A(l=2) + C(l=2)))
1/1,1/2,1/2
1/1,1/3,1/2
=====TEST9, false=======%(A)(+(B)(C))===+(%(A)(B))(%(A)(C))
(A(l=1) + max(B(l=1) , C(l=1)))
max((A(l=1) + B(l=1)) , (A(l=1) + C(l=1)))
1/1,1/2,1/1
1/1,1/3,1/1
=====TEST9, true=======
(A(l=1) + max(B(l=2) , C(l=2)))
max((A(l=2) + B(l=2)) , (A(l=2) + C(l=2)))
1/1,1/2,1/1
1/1,1/3,1/1
=====TEST10, false=======*(A)(B)===*(B)(A)
max(A(l=1) , B(l=1))
max(B(l=1) , A(l=1))
1/1,1/1,1/2
1/1,1/1,1/2
=====TEST10, true=======
max(A(l=2) , B(l=2))
max(B(l=2) , A(l=2))
1/1,1/1,1/2
1/1,1/1,1/2
=====TEST11, false=======+(A)(B)===+(B)(A)
max(A(l=1) , B(l=1))
max(B(l=1) , A(l=1))
1/1,1/1,1/1
1/1,1/1,1/1
=====TEST11, true=======
max(A(l=2) , B(l=2))
```

```
max(B(l=2) , A(l=2))
1/1,1/1,1/1
1/1,1/1,1/1
=====TEST12, false=======*(*(A)(B))(C)===*(A)(*(B)(C))
max(max(A(l=1) , B(l=1)) , C(l=1))
max(A(l=1) , max(B(l=1) , C(l=1)))
1/1,1/1,1/3
1/1,1/1,1/3
=====TEST12, true=======
max(max(A(l=2) , B(l=2)) , C(l=2))
max(A(l=2) , max(B(l=2) , C(l=2)))
1/1,1/1,1/3
1/1,1/1,1/3
=====TEST13, false=======+(+(A)(B))(C)===+(A)(+(B)(C))
max(max(A(l=1) , B(l=1)) , C(l=1))
max(A(l=1) , max(B(l=1) , C(l=1)))
1/1,1/1,1/1
1/1,1/1,1/1
=====TEST13, true=======
max(max(A(l=2) , B(l=2)) , C(l=2))
max(A(l=2) , max(B(l=2) , C(l=2)))
1/1,1/1,1/1
1/1,1/1,1/1
```

Appendix B - test cases for Proposition 9. Test cases for Proposition 9, and associativity and commutativity cases of $\wedge$ (* in the code) and $\vee$ (+ in the code). Some of the lines are very long, and are split in two lines, which is indicated by SP. Test 1 tests $\neg$ reduction 1, Test 2 $\neg$ reduction 2, and so on until Test 400. Test 500 tests $\gtrdot$ reduction 3, Test 8 $\gtrdot$ reduction 4, and Test 9 $\gtrdot$ reduction 5. Test 800 tests obj distribution 1, Test 900 obj distribution 2. Test 10 - 13 test associativity and commutativity of $\wedge$ and $\vee$, which are the same as for Proposition 7.

```
=====TEST1, false=======!s===s
1/4^1+(1/4^1)
1/4^1
1/2,1/1,1/1
1/1,1/1,1/1
=====TEST1, true=======
1/4^1+(1/4^1)
1/4^1
```

```
1/2 ,1/1 ,1/1
1/1 ,1/1 ,1/1
=====TEST2,  false=======!*(A)(B)===+(!A)(!B)
1/4^1+(max(A(l=1),B(l=1)))
max(1/4^1+(A(l=1)),1/4^1+(B(l=1)))
1/2 ,1/1 ,1/2
1/3 ,1/1 ,1/1
=====TEST2,  true=======
1/4^1+(max(A(l=2),B(l=2)))
max(1/4^2+(A(l=2)),1/4^2+(B(l=2)))
1/2 ,1/1 ,1/2
1/3 ,1/1 ,1/1
=====TEST3,  false=======!+(A)(B)===*(!A)(!B)
1/4^1+(max(A(l=1),B(l=1)))
max(1/4^1+(A(l=1)),1/4^1+(B(l=1)))
1/2 ,1/1 ,1/1
1/3 ,1/1 ,1/2
=====TEST3,  true=======
1/4^1+(max(A(l=2),B(l=2)))
max(1/4^2+(A(l=2)),1/4^2+(B(l=2)))
1/2 ,1/1 ,1/1
1/3 ,1/1 ,1/2
=====TEST400,  false=======SP
!%(*(A)(*(B)(C)))(D)===+(+(!A)(+(!B)(!C)))(%(*(A)(*(B)(C)))(!D))
1/4^1+((max(A(l=1),max(B(l=1),C(l=1)))+D(l=1)))
max(max(1/4^1+(A(l=1)),max(1/4^1+(B(l=1)),1/4^1+(C(l=1))))),SP
(max(A(l=1),max(B(l=1),C(l=1)))+1/4^2+(D(l=1))))
1/2 ,1/2 ,1/3
1/5 ,1/2 ,1/3
=====TEST400,  true=======
1/4^1+((max(A(l=2),max(B(l=2),C(l=2)))+D(l=1)))
max(max(1/4^2+(A(l=2)),max(1/4^2+(B(l=2)),1/4^2+(C(l=2))))),(max(A(l=2),SP
max(B(l=2),C(l=2)))+1/4^3+(D(l=2))))
1/2 ,1/2 ,1/3
1/5 ,1/2 ,1/3
=====TEST500,  false=======%(+(A)(+(B)(C)))(D)===+(%(A)(D))(+(%(B)(D))(+(%(C)(D))SP
(+(%(*(A)(B))(D))(+(%(*(A)(C))(D))(+(%(*(B)(C))(D))(%(*(A)(*(B)(C)))(D)))))))
(max(A(l=1),max(B(l=1),C(l=1)))+D(l=1))
max((A(l=1)+D(l=1)),max((B(l=1)+D(l=1)),max((C(l=1)+D(l=1)),SP
max((max(A(l=1),B(l=1))+D(l=1)),max((max(A(l=1),C(l=1))+D(l=1)),SP
```

max((max(B(l=1),C(l=1))+D(l=1)),(max(A(l=1),max(B(l=1),C(l=1)))+D(l=1)))))))

1/1,1/2,1/1

1/1,1/8,1/6

=====TEST500, true=======

(max(A(l=2),max(B(l=2),C(l=2)))+D(l=1))

max((A(l=2)+D(l=2)),max((B(l=2)+D(l=2)),max((C(l=2)+D(l=2)),SP

max((max(A(l=2),B(l=2))+D(l=2)),max((max(A(l=2),C(l=2))+D(l=2)),SP

max((max(B(l=2),C(l=2))+D(l=2)),(max(A(l=2),max(B(l=2),C(l=2)))+D(l=2)))))))

1/1,1/2,1/1

1/1,1/8,1/6

=====TEST8, false=======%(A)(∗(B)(C))===∗(%(A)(B))(%(A)(C))

(A(l=1)+max(B(l=1),C(l=1)))

max((A(l=1)+B(l=1)),(A(l=1)+C(l=1)))

1/1,1/2,1/2

1/1,1/3,1/2

=====TEST8, true=======

(A(l=1)+max(B(l=2),C(l=2)))

max((A(l=2)+B(l=2)),(A(l=2)+C(l=2)))

1/1,1/2,1/2

1/1,1/3,1/2

=====TEST9, false=======%(A)(+(B)(C))===+(%(A)(B))(%(A)(C))

(A(l=1)+max(B(l=1),C(l=1)))

max((A(l=1)+B(l=1)),(A(l=1)+C(l=1)))

1/1,1/2,1/1

1/1,1/3,1/1

=====TEST9, true=======

(A(l=1)+max(B(l=2),C(l=2)))

max((A(l=2)+B(l=2)),(A(l=2)+C(l=2)))

1/1,1/2,1/1

1/1,1/3,1/1

=====TEST800, false=======%(∗(A)(+(B)(C)))(D)===%(+(∗(A)(B))(∗(A)(C)))(D)

(max(A(l=1),max(B(l=1),C(l=1)))+D(l=1))

(max(max(A(l=1),B(l=1)),max(A(l=1),C(l=1)))+D(l=1))

1/1,1/2,1/2

1/1,1/2,1/3

=====TEST800, true=======

(max(A(l=2),max(B(l=2),C(l=2)))+D(l=1))

(max(max(A(l=2),B(l=2)),max(A(l=2),C(l=2)))+D(l=1))

1/1,1/2,1/2

1/1,1/2,1/3

```
=====TEST900, false=======%(*(+(A)(B))(C))(D)===%(+(*(A)(C))(*(B)(C)))(D)
(max(max(A(l=1),B(l=1)),C(l=1))+D(l=1))
(max(max(A(l=1),C(l=1)),max(B(l=1),C(l=1)))+D(l=1))
1/1,1/2,1/2
1/1,1/2,1/3
=====TEST900, true=======
(max(max(A(l=2),B(l=2)),C(l=2))+D(l=1))
(max(max(A(l=2),C(l=2)),max(B(l=2),C(l=2)))+D(l=1))
1/1,1/2,1/2
1/1,1/2,1/3
=====TEST10, false=======*(A)(B)===*(B)(A)
max(A(l=1),B(l=1))
max(B(l=1),A(l=1))
1/1,1/1,1/2
1/1,1/1,1/2
=====TEST10, true=======
max(A(l=2),B(l=2))
max(B(l=2),A(l=2))
1/1,1/1,1/2
1/1,1/1,1/2
=====TEST11, false=======+(A)(B)===+(B)(A)
max(A(l=1),B(l=1))
max(B(l=1),A(l=1))
1/1,1/1,1/1
1/1,1/1,1/1
=====TEST11, true=======
max(A(l=2),B(l=2))
max(B(l=2),A(l=2))
1/1,1/1,1/1
1/1,1/1,1/1
=====TEST12, false=======*(*(A)(B))(C)===*(A)(*(B)(C))
max(max(A(l=1),B(l=1)),C(l=1))
max(A(l=1),max(B(l=1),C(l=1)))
1/1,1/1,1/3
1/1,1/1,1/3
=====TEST12, true=======
max(max(A(l=2),B(l=2)),C(l=2))
max(A(l=2),max(B(l=2),C(l=2)))
1/1,1/1,1/3
1/1,1/1,1/3
```

```
=====TEST13, false=======+(+(A)(B))(C)===+(A)(+(B)(C))
max(max(A(l=1),B(l=1)),C(l=1))
max(A(l=1),max(B(l=1),C(l=1)))
1/1,1/1,1/1
1/1,1/1,1/1
=====TEST13, true=======
max(max(A(l=2),B(l=2)),C(l=2))
max(A(l=2),max(B(l=2),C(l=2)))
1/1,1/1,1/1
1/1,1/1,1/1
```